CHAPTER 3 DIFFERENTIATION

3.1 TANGENTS AND THE DERIVATIVE AT A POINT

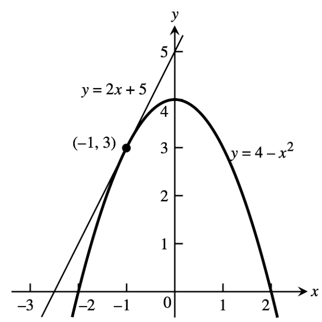
1. P_1: $m_1 = 1$, P_2: $m_2 = 5$

2. P_1: $m_1 = -2$, P_2: $m_2 = 0$

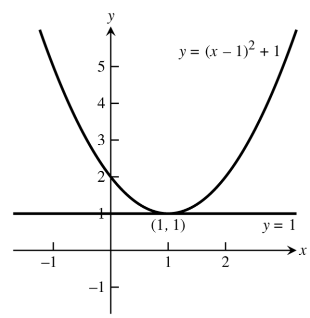
3. P_1: $m_1 = \frac{5}{2}$, P_2: $m_2 = -\frac{1}{2}$

4. P_1: $m_1 = 3$, P_2: $m_2 = -3$

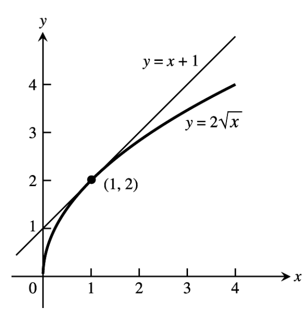
5. $m = \lim\limits_{h \to 0} \frac{[4-(-1+h)^2]-(4-(-1)^2)}{h}$
$= \lim\limits_{h \to 0} \frac{-(1-2h+h^2)+1}{h} = \lim\limits_{h \to 0} \frac{h(2-h)}{h} = 2;$
at $(-1, 3)$: $y = 3 + 2(x-(-1)) \Rightarrow y = 2x + 5$, tangent line



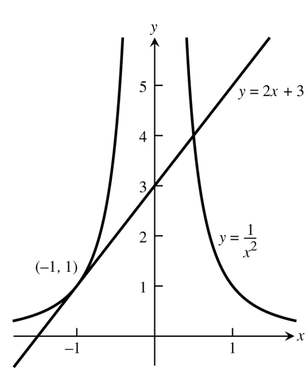
6. $m = \lim\limits_{h \to 0} \frac{[(1+h-1)^2+1]-[(1-1)^2+1]}{h} = \lim\limits_{h \to 0} \frac{h^2}{h}$
$= \lim\limits_{h \to 0} h = 0$; at $(1, 1)$: $y = 1 + 0(x-1) \Rightarrow y = 1$, tangent line



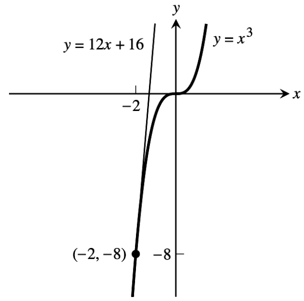
7. $m = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2}{h} \cdot \frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$
$= \lim\limits_{h \to 0} \frac{4(1+h)-4}{2h\left(\sqrt{1+h}+1\right)} = \lim\limits_{h \to 0} \frac{2}{\sqrt{1+h}+1} = 1;$
at $(1, 2)$: $y = 2 + 1(x-1) \Rightarrow y = x + 1$, tangent line



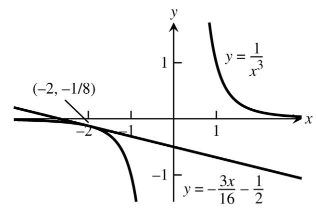
8. $m = \lim\limits_{h \to 0} \frac{\frac{1}{(-1+h)^2}-\frac{1}{(-1)^2}}{h} = \lim\limits_{h \to 0} \frac{1-(-1+h)^2}{h(-1+h)^2}$
$= \lim\limits_{h \to 0} \frac{-(-2h+h^2)}{h(-1+h)^2} = \lim\limits_{h \to 0} \frac{2-h}{(-1+h)^2} = 2;$
at $(-1, 1)$: $y = 1 + 2(x-(-1)) \Rightarrow y = 2x + 3$, tangent line



9. $m = \lim\limits_{h \to 0} \frac{(-2+h)^3-(-2)^3}{h} = \lim\limits_{h \to 0} \frac{-8+12h-6h^2+h^3+8}{h}$
$= \lim\limits_{h \to 0} (12-6h+h^2) = 12;$
at $(-2,-8)$: $y = -8+12(x-(-2)) \Rightarrow y = 12x+16$, tangent line



10. $m = \lim\limits_{h \to 0} \frac{\frac{1}{(-2+h)^3}-\frac{1}{(-2)^3}}{h} = \lim\limits_{h \to 0} \frac{-8-(-2+h)^3}{-8h(-2+h)^3}$
$= \lim\limits_{h \to 0} \frac{-(12h-6h^2+h^3)}{-8h(-2+h)^3} = \lim\limits_{h \to 0} \frac{12-6h+h^2}{8(-2+h)^3}$
$= \frac{12}{8(-8)} = -\frac{3}{16};$
at $\left(-2,-\frac{1}{8}\right)$: $y = -\frac{1}{8}-\frac{3}{16}(x-(-2))$
$\Rightarrow y = -\frac{3}{16}x-\frac{1}{2}$, tangent line



11. $m = \lim\limits_{h \to 0} \frac{[(2+h)^2+1]-5}{h} = \lim\limits_{h \to 0} \frac{(5+4h+h^2)-5}{h} = \lim\limits_{h \to 0} \frac{h(4+h)}{h} = 4;$
at $(2,5)$: $y-5 = 4(x-2)$, tangent line

12. $m = \lim\limits_{h \to 0} \frac{[(1+h)-2(1+h)^2]-(-1)}{h} = \lim\limits_{h \to 0} \frac{(1+h-2-4h-2h^2)+1}{h} = \lim\limits_{h \to 0} \frac{h(-3-2h)}{h} = -3;$
at $(1,-1)$: $y+1 = -3(x-1)$, tangent line

13. $m = \lim\limits_{h \to 0} \frac{\frac{3+h}{(3+h)-2}-3}{h} = \lim\limits_{h \to 0} \frac{(3+h)-3(h+1)}{h(h+1)} = \lim\limits_{h \to 0} \frac{-2h}{h(h+1)} = -2;$
at $(3,3)$: $y-3 = -2(x-3)$, tangent line

14. $m = \lim\limits_{h \to 0} \frac{\frac{8}{(2+h)^2}-2}{h} = \lim\limits_{h \to 0} \frac{8-2(2+h)^2}{h(2+h)^2} = \lim\limits_{h \to 0} \frac{8-2(4+4h+h^2)}{h(2+h)^2} = \lim\limits_{h \to 0} \frac{-2h(4+h)}{h(2+h)^2} = \frac{-8}{4} = -2;$
at $(2,2)$: $y-2 = -2(x-2)$

15. $m = \lim\limits_{h \to 0} \frac{(2+h)^3-8}{h} = \lim\limits_{h \to 0} \frac{(8+12h+6h^2+h^3)-8}{h} = \lim\limits_{h \to 0} \frac{h(12+6h+h^2)}{h} = 12;$
at $(2,8)$: $y-8 = 12(t-2)$, tangent line

16. $m = \lim\limits_{h \to 0} \frac{[(1+h)^3+3(1+h)]-4}{h} = \lim\limits_{h \to 0} \frac{(1+3h+3h^2+h^3+3+3h)-4}{h} = \lim\limits_{h \to 0} \frac{h(6+3h+h^2)}{h} = 6;$
at $(1,4)$: $y-4 = 6(t-1)$, tangent line

17. $m = \lim\limits_{h \to 0} \frac{\sqrt{4+h}-2}{h} = \lim\limits_{h \to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim\limits_{h \to 0} \frac{(4+h)-4}{h\left(\sqrt{4+h}+2\right)} = \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{4+h}+2\right)} = \frac{1}{\sqrt{4}+2}$
$= \frac{1}{4}$; at $(4,2)$: $y-2 = \frac{1}{4}(x-4)$, tangent line

18. $m = \lim\limits_{h \to 0} \frac{\sqrt{(8+h)+1}-3}{h} = \lim\limits_{h \to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim\limits_{h \to 0} \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)} = \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{9+h}+3\right)}$
$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$; at $(8,3)$: $y-3 = \frac{1}{6}(x-8)$, tangent line

19. At $x=-1$, $y=5 \Rightarrow m = \lim\limits_{h \to 0} \frac{5(-1+h)^2-5}{h} = \lim\limits_{h \to 0} \frac{5(1-2h+h^2)-5}{h} = \lim\limits_{h \to 0} \frac{5h(-2+h)}{h} = -10$, slope

20. At $x = 2$, $y = -3 \Rightarrow m = \lim_{h \to 0} \frac{[1-(2+h)^2]-(-3)}{h} = \lim_{h \to 0} \frac{(1-4-4h-h^2)+3}{h} = \lim_{h \to 0} \frac{-h(4+h)}{h} = -4$, slope

21. At $x = 3$, $y = \frac{1}{2} \Rightarrow m = \lim_{h \to 0} \frac{\frac{1}{(3+h)-1} - \frac{1}{2}}{h} = \lim_{h \to 0} \frac{2-(2+h)}{2h(2+h)} = \lim_{h \to 0} \frac{-h}{2h(2+h)} = -\frac{1}{4}$, slope

22. At $x = 0$, $y = -1 \Rightarrow m = \lim_{h \to 0} \frac{\frac{h-1}{h+1} - (-1)}{h} = \lim_{h \to 0} \frac{(h-1)+(h+1)}{h(h+1)} = \lim_{h \to 0} \frac{2h}{h(h+1)} = 2$, slope

23. At a horizontal tangent the slope $m = 0 \Rightarrow 0 = m = \lim_{h \to 0} \frac{[(x+h)^2+4(x+h)-1]-(x^2+4x-1)}{h}$
$= \lim_{h \to 0} \frac{(x^2+2xh+h^2+4x+4h-1)-(x^2+4x-1)}{h} = \lim_{h \to 0} \frac{(2xh+h^2+4h)}{h} = \lim_{h \to 0} (2x+h+4) = 2x+4$;
$2x+4 = 0 \Rightarrow x = -2$. Then $f(-2) = 4-8-1 = -5 \Rightarrow (-2,-5)$ is the point on the graph where there is a horizontal tangent.

24. $0 = m = \lim_{h \to 0} \frac{[(x+h)^3-3(x+h)]-(x^3-3x)}{h} = \lim_{h \to 0} \frac{(x^3+3x^2h+3xh^2+h^3-3x-3h)-(x^3-3x)}{h}$
$= \lim_{h \to 0} \frac{3x^2h+3xh^2+h^3-3h}{h} = \lim_{h \to 0} (3x^2+3xh+h^2-3) = 3x^2-3$; $3x^2-3 = 0 \Rightarrow x = -1$ or $x = 1$. Then $f(-1) = 2$ and $f(1) = -2 \Rightarrow (-1,2)$ and $(1,-2)$ are the points on the graph where a horizontal tangent exists.

25. $-1 = m = \lim_{h \to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h} = \lim_{h \to 0} \frac{(x-1)-(x+h-1)}{h(x-1)(x+h-1)} = \lim_{h \to 0} \frac{-h}{h(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$
$\Rightarrow (x-1)^2 = 1 \Rightarrow x^2-2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. If $x = 0$, then $y = -1$ and $m = -1$
$\Rightarrow y = -1-(x-0) = -(x+1)$. If $x = 2$, then $y = 1$ and $m = -1 \Rightarrow y = 1-(x-2) = -(x-3)$.

26. $\frac{1}{4} = m = \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \lim_{h \to 0} \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$
$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$. Thus, $\frac{1}{4} = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4 \Rightarrow y = 2$. The tangent line is $y = 2+\frac{1}{4}(x-4) = \frac{x}{4}+1$.

27. $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h \to 0} \frac{(100-4.9(2+h)^2)-(100-4.9(2)^2)}{h} = \lim_{h \to 0} \frac{-4.9\,(4+4h+h^2)+4.9(4)}{h}$
$= \lim_{h \to 0} (-19.6-4.9h) = -19.6$. The minus sign indicates the object is falling downward at a speed of 19.6 m/sec.

28. $\lim_{h \to 0} \frac{f(10+h)-f(10)}{h} = \lim_{h \to 0} \frac{3(10+h)^2-3(10)^2}{h} = \lim_{h \to 0} \frac{3\,(20h+h^2)}{h} = 60$ ft/sec.

29. $\lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h \to 0} \frac{\pi(3+h)^2-\pi(3)^2}{h} = \lim_{h \to 0} \frac{\pi\,[9+6h+h^2-9]}{h} = \lim_{h \to 0} \pi(6+h) = 6\pi$

30. $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h \to 0} \frac{\frac{4\pi}{3}(2+h)^3-\frac{4\pi}{3}(2)^3}{h} = \lim_{h \to 0} \frac{\frac{4\pi}{3}[12h+6h^2+h^3]}{h} = \lim_{h \to 0} \frac{4\pi}{3}[12+6h+h^2] = 16\pi$

31. At (x_0, mx_0+b) the slope of the tangent line is $\lim_{h \to 0} \frac{(m(x_0+h)+b)-(m\,x_0+b)}{(x_0+h)-x_0} = \lim_{h \to 0} \frac{m\,h}{h} = \lim_{h \to 0} m = m$.
The equation of the tangent line is $y-(m\,x_0+b) = m(x-x_0) \Rightarrow y = mx+b$.

32. At $x = 4$, $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$ and $m = \lim_{h \to 0} \frac{\frac{1}{\sqrt{4+h}}-\frac{1}{2}}{h} = \lim_{h \to 0} \left[\frac{\frac{1}{\sqrt{4+h}}-\frac{1}{2}}{h} \cdot \frac{2\sqrt{4+h}}{2\sqrt{4+h}}\right] = \lim_{h \to 0} \left(\frac{2-\sqrt{4+h}}{2h\sqrt{4+h}}\right)$
$= \lim_{h \to 0} \left[\frac{2-\sqrt{4+h}}{2h\sqrt{4+h}} \cdot \frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}\right] = \lim_{h \to 0} \left(\frac{4-(4+h)}{2h\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right) = \lim_{h \to 0} \left(\frac{-h}{2h\sqrt{4+h}\left(2+\sqrt{4+h}\right)}\right)$

$$= \lim_{h \to 0} \left(\frac{-1}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)} \right) = -\frac{1}{2\sqrt{4}\left(2+\sqrt{4}\right)} = -\frac{1}{16}$$

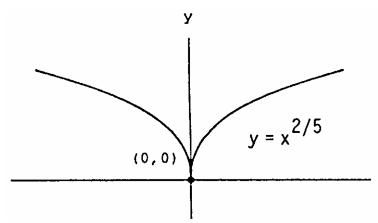
33. Slope at origin $= \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, f(x) does have a tangent at the origin with slope 0.

34. $\lim_{h \to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} \sin \frac{1}{h}$. Since $\lim_{h \to 0} \sin \frac{1}{h}$ does not exist, f(x) has no tangent at the origin.

35. $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{-1-0}{h} = \infty$, and $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{1-0}{h} = \infty$. Therefore, $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \infty \Rightarrow$ yes, the graph of f has a vertical tangent at the origin.

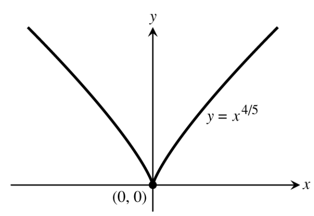
36. $\lim_{h \to 0^-} \frac{U(0+h)-U(0)}{h} = \lim_{h \to 0^-} \frac{0-1}{h} = \infty$, and $\lim_{h \to 0^+} \frac{U(0+h)-U(0)}{h} = \lim_{h \to 0^+} \frac{1-1}{h} = 0 \Rightarrow$ no, the graph of f does not have a vertical tangent at (0, 1) because the limit does not exist.

37. (a) The graph appears to have a cusp at x = 0.



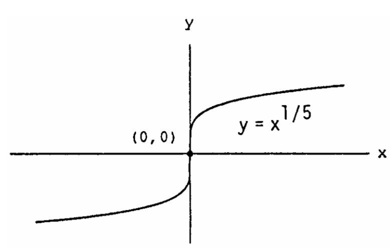
(b) $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{h^{2/5}-0}{h} = \lim_{h \to 0^-} \frac{1}{h^{3/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{3/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = x^{2/5}$ does not have a vertical tangent at x = 0.

38. (a) The graph appears to have a cusp at x = 0.



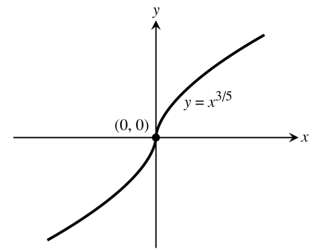
(b) $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{h^{4/5}-0}{h} = \lim_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{1/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow y = x^{4/5}$ does not have a vertical tangent at x = 0.

39. (a) The graph appears to have a vertical tangent at x = 0.



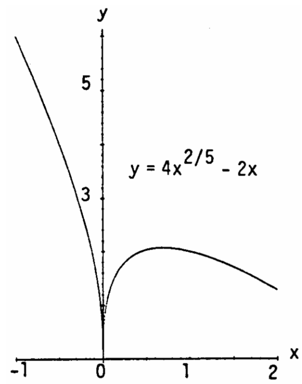
(b) $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^{1/5}-0}{h} = \lim_{h \to 0} \frac{1}{h^{4/5}} = \infty \Rightarrow y = x^{1/5}$ has a vertical tangent at x = 0.

40. (a) The graph appears to have a vertical tangent at $x = 0$.



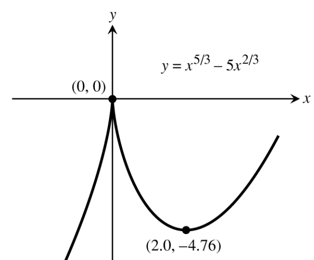
(b) $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{3/5} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}} = \infty \Rightarrow$ the graph of $y = x^{3/5}$ has a vertical tangent at $x = 0$.

41. (a) The graph appears to have a cusp at $x = 0$.



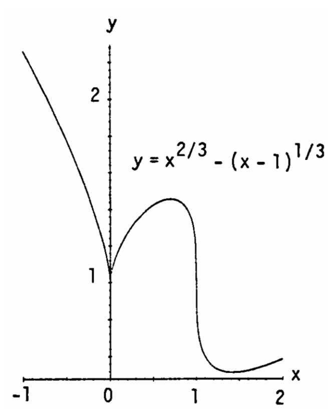
(b) $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{4h^{2/5} - 2h}{h} = \lim_{h \to 0^-} \frac{4}{h^{3/5}} - 2 = -\infty$ and $\lim_{h \to 0^+} \frac{4}{h^{3/5}} - 2 = \infty$
$\Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = 4x^{2/5} - 2x$ does not have a vertical tangent at $x = 0$.

42. (a) The graph appears to have a cusp at $x = 0$.



(b) $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{5/3} - 5h^{2/3}}{h} = \lim_{h \to 0} h^{2/3} - \frac{5}{h^{1/3}} = 0 - \lim_{h \to 0} \frac{5}{h^{1/3}}$ does not exist $\Rightarrow$ the graph of $y = x^{5/3} - 5x^{2/3}$ does not have a vertical tangent at $x = 0$.

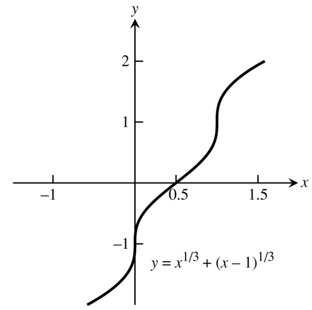
43. (a) The graph appears to have a vertical tangent at $x = 1$ and a cusp at $x = 0$.



(b) $x = 1$: $\lim_{h \to 0} \frac{(1+h)^{2/3} - (1+h-1)^{1/3} - 1}{h} = \lim_{h \to 0} \frac{(1+h)^{2/3} - h^{1/3} - 1}{h} = -\infty$
$\Rightarrow y = x^{2/3} - (x-1)^{1/3}$ has a vertical tangent at $x = 1$;

x = 0: $\lim\limits_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{2/3}-(h-1)^{1/3}-(-1)^{1/3}}{h} = \lim\limits_{h \to 0} \left[\frac{1}{h^{1/3}} - \frac{(h-1)^{1/3}}{h} + \frac{1}{h}\right]$ does not exist $\Rightarrow$ $y = x^{2/3} - (x-1)^{1/3}$ does not have a vertical tangent at x = 0.

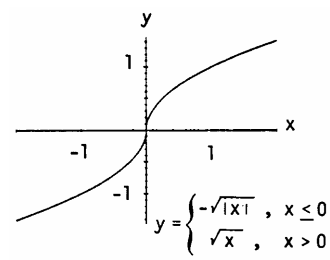
44. (a) The graph appears to have vertical tangents at x = 0 and x = 1.



(b) x = 0: $\lim\limits_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{1/3}+(h-1)^{1/3}-(-1)^{1/3}}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at x = 0;

x = 1: $\lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h \to 0} \frac{(1+h)^{1/3}+(1+h-1)^{1/3}-1}{h} = \infty \Rightarrow y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at x = 1.

45. (a) The graph appears to have a vertical tangent at x = 0.

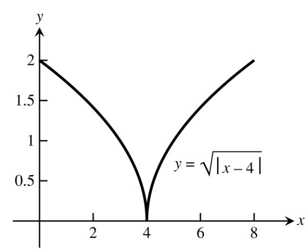


(b) $\lim\limits_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{x \to 0^+} \frac{\sqrt{h}-0}{h} = \lim\limits_{h \to 0} \frac{1}{\sqrt{h}} = \infty$;

$\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^-} \frac{-\sqrt{|h|}-0}{h} = \lim\limits_{h \to 0^-} \frac{-\sqrt{|h|}}{-|h|} = \lim\limits_{h \to 0^-} \frac{1}{\sqrt{|h|}} = \infty$

$\Rightarrow$ y has a vertical tangent at x = 0.

46. (a) The graph appears to have a cusp at x = 4.



(b) $\lim\limits_{h \to 0^+} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|4-(4+h)|}-0}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|h|}}{h} = \lim\limits_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty$;

$\lim\limits_{h \to 0^-} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|4-(4+h)|}}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|h|}}{-|h|} = \lim\limits_{h \to 0^-} \frac{-1}{\sqrt{|h|}} = -\infty$

$\Rightarrow$ $y = \sqrt{4-x}$ does not have a vertical tangent at x = 4.

47-50. Example CAS commands:

<u>Maple</u>:

```
f := x -> x^3 + 2*x;x0 := 0;
plot( f(x), x=x0-1/2..x0+3, color=black,              # part (a)
      title="Section 3.1, #47(a)" );
q := unapply( (f(x0+h)-f(x0))/h, h );                  # part (b)
```

```
L := limit( q(h), h=0 );                              # part (c)
sec_lines := seq( f(x0)+q(h)*(x-x0), h=1..3 );        # part (d)
tan_line := f(x0) + L*(x-x0);
plot( [f(x),tan_line,sec_lines], x=x0-1/2..x0+3, color=black,
      linestyle=[1,2,5,6,7], title="Section 3.1, #47(d)",
      legend=["y=f(x)","Tangent line at x=0","Secant line (h=1)",
              "Secant line (h=2)","Secant line (h=3)"] );
```

Mathematica: (function and value for x0 may change)

```
Clear[f, m, x, h]
x0 = p;
f[x_]: = Cos[x]  +  4Sin[2x]
Plot[f[x], {x, x0 − 1, x0 + 3}]
dq[h_]: = (f[x0+h]  − f[x0])/h
m = Limit[dq[h], h  →  0]
ytan: = f[x0] + m(x − x0)
y1: =  f[x0] + dq[1](x − x0)
y2: =  f[x0] + dq[2](x − x0)
y3: =  f[x0] + dq[3](x − x0)
Plot[{f[x], ytan, y1, y2, y3}, {x, x0 − 1, x0 + 3}]
```

3.2 THE DERIVATIVE AS A FUNCTION

1. Step 1: $f(x) = 4 - x^2$ and $f(x+h) = 4 - (x+h)^2$

   Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{[4-(x+h)^2]-(4-x^2)}{h} = \frac{(4-x^2-2xh-h^2)-4+x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h}$
   $= -2x - h$

   Step 3: $f'(x) = \lim_{h\to 0} (-2x-h) = -2x$; $f'(-3) = 6$, $f'(0) = 0$, $f'(1) = -2$

2. $F(x) = (x-1)^2 + 1$ and $F(x+h) = (x+h-1)^2 + 1 \Rightarrow F'(x) = \lim_{h\to 0} \frac{[(x+h-1)^2+1]-[(x-1)^2+1]}{h}$
   $= \lim_{h\to 0} \frac{(x^2+2xh+h^2-2x-2h+1+1)-(x^2-2x+1+1)}{h} = \lim_{h\to 0} \frac{2xh+h^2-2h}{h} = \lim_{h\to 0} (2x+h-2)$
   $= 2(x-1)$; $F'(-1) = -4$, $F'(0) = -2$, $F'(2) = 2$

3. Step 1: $g(t) = \frac{1}{t^2}$ and $g(t+h) = \frac{1}{(t+h)^2}$

   Step 2: $\frac{g(t+h)-g(t)}{h} = \frac{\frac{1}{(t+h)^2}-\frac{1}{t^2}}{h} = \frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2\cdot t^2}\right)}{h} = \frac{t^2-(t^2+2th+h^2)}{(t+h)^2\cdot t^2\cdot h} = \frac{-2th-h^2}{(t+h)^2\,t^2h}$
   $= \frac{h(-2t-h)}{(t+h)^2\,t^2h} = \frac{-2t-h}{(t+h)^2\,t^2}$

   Step 3: $g'(t) = \lim_{h\to 0} \frac{-2t-h}{(t+h)^2\,t^2} = \frac{-2t}{t^2\cdot t^2} = \frac{-2}{t^3}$; $g'(-1) = 2$, $g'(2) = -\frac{1}{4}$, $g'\left(\sqrt{3}\right) = -\frac{2}{3\sqrt{3}}$

4. $k(z) = \frac{1-z}{2z}$ and $k(z+h) = \frac{1-(z+h)}{2(z+h)} \Rightarrow k'(z) = \lim_{h\to 0} \frac{\left(\frac{1-(z+h)}{2(z+h)} - \frac{1-z}{2z}\right)}{h}$
   $= \lim_{h\to 0} \frac{(1-z-h)z-(1-z)(z+h)}{2(z+h)zh} = \lim_{h\to 0} \frac{z-z^2-zh-z-h+z^2+zh}{2(z+h)zh} = \lim_{h\to 0} \frac{-h}{2(z+h)zh} = \lim_{h\to 0} \frac{-1}{2(z+h)z}$
   $= \frac{-1}{2z^2}$; $k'(-1) = -\frac{1}{2}$, $k'(1) = -\frac{1}{2}$, $k'\left(\sqrt{2}\right) = -\frac{1}{4}$

5. Step 1: $p(\theta) = \sqrt{3\theta}$ and $p(\theta+h) = \sqrt{3(\theta+h)}$

Step 2: $\frac{p(\theta+h)-p(\theta)}{h} = \frac{\sqrt{3(\theta+h)}-\sqrt{3\theta}}{h} = \frac{\left(\sqrt{3\theta+3h}-\sqrt{3\theta}\right)}{h} \cdot \frac{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)} = \frac{(3\theta+3h)-3\theta}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}$

$= \frac{3h}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)} = \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}}$

Step 3: $p'(\theta) = \lim\limits_{h\to 0} \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}} = \frac{3}{\sqrt{3\theta}+\sqrt{3\theta}} = \frac{3}{2\sqrt{3\theta}}$; $p'(1) = \frac{3}{2\sqrt{3}}$, $p'(3) = \frac{1}{2}$, $p'\left(\frac{2}{3}\right) = \frac{3}{2\sqrt{2}}$

6. $r(s) = \sqrt{2s+1}$ and $r(s+h) = \sqrt{2(s+h)+1} \Rightarrow r'(s) = \lim\limits_{h\to 0} \frac{\sqrt{2s+2h+1}-\sqrt{2s+1}}{h}$

$= \lim\limits_{h\to 0} \frac{\left(\sqrt{2s+h+1}-\sqrt{2s+1}\right)}{h} \cdot \frac{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)} = \lim\limits_{h\to 0} \frac{(2s+2h+1)-(2s+1)}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}$

$= \lim\limits_{h\to 0} \frac{2h}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)} = \lim\limits_{h\to 0} \frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}} = \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}} = \frac{2}{2\sqrt{2s+1}}$

$= \frac{1}{\sqrt{2s+1}}$; $r'(0) = 1$, $r'(1) = \frac{1}{\sqrt{3}}$, $r'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$

7. $y = f(x) = 2x^3$ and $f(x+h) = 2(x+h)^3 \Rightarrow \frac{dy}{dx} = \lim\limits_{h\to 0} \frac{2(x+h)^3-2x^3}{h} = \lim\limits_{h\to 0} \frac{2\left(x^3+3x^2h+3xh^2+h^3\right)-2x^3}{h}$

$= \lim\limits_{h\to 0} \frac{6x^2h+6xh^2+2h^3}{h} = \lim\limits_{h\to 0} \frac{h\left(6x^2+6xh+2h^2\right)}{h} = \lim\limits_{h\to 0} \left(6x^2+6xh+2h^2\right) = 6x^2$

8. $r = s^3-2s^2+3 \Rightarrow \frac{dr}{ds} = \lim\limits_{h\to 0} \frac{\left((s+h)^3-2(s+h)^2+3\right)-\left(s^3-2s^2+3\right)}{h} = \lim\limits_{h\to 0} \frac{s^3+3s^2h+3sh^2+h^3-2s^2-4sh-h^2+3-s^3+2s^2-3}{h}$

$= \lim\limits_{h\to 0} \frac{3s^2h+3sh^2+h^3-4sh-h^2}{h} = \lim\limits_{h\to 0} \frac{h\left(3s^2+3sh+h^2-4s-h\right)}{h} = \lim\limits_{h\to 0} \left(3s^2+3sh+h^2-4s-h\right) = 3s^2-2s$

9. $s = r(t) = \frac{t}{2t+1}$ and $r(t+h) = \frac{t+h}{2(t+h)+1} \Rightarrow \frac{ds}{dt} = \lim\limits_{h\to 0} \frac{\left(\frac{t+h}{2(t+h)+1}\right)-\left(\frac{t}{2t+1}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{\left(\frac{(t+h)(2t+1)-t(2t+2h+1)}{(2t+2h+1)(2t+1)}\right)}{h} = \lim\limits_{h\to 0} \frac{(t+h)(2t+1)-t(2t+2h+1)}{(2t+2h+1)(2t+1)h}$

$= \lim\limits_{h\to 0} \frac{2t^2+t+2ht+h-2t^2-2ht-t}{(2t+2h+1)(2t+1)h} = \lim\limits_{h\to 0} \frac{h}{(2t+2h+1)(2t+1)h} = \lim\limits_{h\to 0} \frac{1}{(2t+2h+1)(2t+1)}$

$= \frac{1}{(2t+1)(2t+1)} = \frac{1}{(2t+1)^2}$

10. $\frac{dv}{dt} = \lim\limits_{h\to 0} \frac{\left[(t+h)-\frac{1}{t+h}\right]-\left(t-\frac{1}{t}\right)}{h} = \lim\limits_{h\to 0} \frac{h-\frac{1}{t+h}+\frac{1}{t}}{h} = \lim\limits_{h\to 0} \frac{\left(\frac{h(t+h)t-t+(t+h)}{(t+h)t}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{ht^2+h^2t+h}{h(t+h)t} = \lim\limits_{h\to 0} \frac{t^2+ht+1}{(t+h)t} = \frac{t^2+1}{t^2} = 1+\frac{1}{t^2}$

11. $p = f(q) = \frac{1}{\sqrt{q+1}}$ and $f(q+h) = \frac{1}{\sqrt{(q+h)+1}} \Rightarrow \frac{dp}{dq} = \lim\limits_{h\to 0} \frac{\left(\frac{1}{\sqrt{(q+h)+1}}\right)-\left(\frac{1}{\sqrt{q+1}}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{\left(\frac{\sqrt{q+1}-\sqrt{q+h+1}}{\sqrt{q+h+1}\sqrt{q+1}}\right)}{h} = \lim\limits_{h\to 0} \frac{\sqrt{q+1}-\sqrt{q+h+1}}{h\sqrt{q+h+1}\sqrt{q+1}}$

$= \lim\limits_{h\to 0} \frac{\left(\sqrt{q+1}-\sqrt{q+h+1}\right)}{h\sqrt{q+h+1}\sqrt{q+1}} \cdot \frac{\left(\sqrt{q+1}+\sqrt{q+h+1}\right)}{\left(\sqrt{q+1}+\sqrt{q+h+1}\right)} = \lim\limits_{h\to 0} \frac{(q+1)-(q+h+1)}{h\sqrt{q+h+1}\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+h+1}\right)}$

$= \lim\limits_{h\to 0} \frac{-h}{h\sqrt{q+h+1}\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+h+1}\right)} = \lim\limits_{h\to 0} \frac{-1}{\sqrt{q+h+1}\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+h+1}\right)}$

$= \frac{-1}{\sqrt{q+1}\sqrt{q+1}\left(\sqrt{q+1}+\sqrt{q+1}\right)} = \frac{-1}{2(q+1)\sqrt{q+1}}$

12. $\frac{dz}{dw} = \lim\limits_{h\to 0} \frac{\left(\frac{1}{\sqrt{3(w+h)-2}}-\frac{1}{\sqrt{3w-2}}\right)}{h} = \lim\limits_{h\to 0} \frac{\sqrt{3w-2}-\sqrt{3w+3h-2}}{h\sqrt{3w+3h-2}\sqrt{3w-2}}$

$= \lim\limits_{h\to 0} \frac{\left(\sqrt{3w-2}-\sqrt{3w+3h-2}\right)}{h\sqrt{3w+3h-2}\sqrt{3w-2}} \cdot \frac{\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)}{\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)} = \lim\limits_{h\to 0} \frac{(3w-2)-(3w+3h-2)}{h\sqrt{3w+3h-2}\sqrt{3w-2}\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)}$

$= \lim\limits_{h\to 0} \frac{-3}{\sqrt{3w+3h-2}\sqrt{3w-2}\left(\sqrt{3w-2}+\sqrt{3w+3h-2}\right)} = \frac{-3}{\sqrt{3w-2}\sqrt{3w-2}\left(\sqrt{3w-2}+\sqrt{3w-2}\right)}$

$= \frac{-3}{2(3w-2)\sqrt{3w-2}}$

13. $f(x) = x + \frac{9}{x}$ and $f(x+h) = (x+h) + \frac{9}{(x+h)} \Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{\left[(x+h)+\frac{9}{(x+h)}\right]-\left[x+\frac{9}{x}\right]}{h}$

$= \frac{x(x+h)^2+9x-x^2(x+h)-9(x+h)}{x(x+h)h} = \frac{x^3+2x^2h+xh^2+9x-x^3-x^2h-9x-9h}{x(x+h)h} = \frac{x^2h+xh^2-9h}{x(x+h)h}$

$= \frac{h(x^2+xh-9)}{x(x+h)h} = \frac{x^2+xh-9}{x(x+h)}$; $f'(x) = \lim\limits_{h\to 0} \frac{x^2+xh-9}{x(x+h)} = \frac{x^2-9}{x^2} = 1 - \frac{9}{x^2}$; $m = f'(-3) = 0$

14. $k(x) = \frac{1}{2+x}$ and $k(x+h) = \frac{1}{2+(x+h)} \Rightarrow k'(x) = \lim\limits_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim\limits_{h\to 0} \frac{\left(\frac{1}{2+x+h}-\frac{1}{2+x}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{(2+x)-(2+x+h)}{h(2+x)(2+x+h)} = \lim\limits_{h\to 0} \frac{-h}{h(2+x)(2+x+h)} = \lim\limits_{h\to 0} \frac{-1}{(2+x)(2+x+h)} = \frac{-1}{(2+x)^2}$;

$k'(2) = -\frac{1}{16}$

15. $\frac{ds}{dt} = \lim\limits_{h\to 0} \frac{[(t+h)^3-(t+h)^2]-(t^3-t^2)}{h} = \lim\limits_{h\to 0} \frac{(t^3+3t^2h+3th^2+h^3)-(t^2+2th+h^2)-t^3+t^2}{h}$

$= \lim\limits_{h\to 0} \frac{3t^2h+3th^2+h^3-2th-h^2}{h} = \lim\limits_{h\to 0} \frac{h(3t^2+3th+h^2-2t-h)}{h} = \lim\limits_{h\to 0} (3t^2+3th+h^2-2t-h)$

$= 3t^2-2t$; $m = \left.\frac{ds}{dt}\right|_{t=-1} = 5$

16. $\frac{dy}{dx} = \lim\limits_{h\to 0} \frac{\frac{(x+h)+3}{1-(x+h)}-\frac{x+3}{1-x}}{h} = \lim\limits_{h\to 0} \frac{\frac{(x+h+3)(1-x)-(x+3)(1-x-h)}{(1-x-h)(1-x)}}{h} = \lim\limits_{h\to 0} \frac{x+h+3-x^2-xh-3x-x-3+x^2+3x+xh+3h}{h(1-x-h)(1-x)}$

$= \lim\limits_{h\to 0} \frac{4h}{h(1-x-h)(1-x)} = \lim\limits_{h\to 0} \frac{4}{(1-x-h)(1-x)} = \frac{4}{(1-x)^2}$; $\left.\frac{dy}{dx}\right|_{x=-2} = \frac{4}{(3)^2} = \frac{4}{9}$

17. $f(x) = \frac{8}{\sqrt{x-2}}$ and $f(x+h) = \frac{8}{\sqrt{(x+h)-2}} \Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{\frac{8}{\sqrt{(x+h)-2}}-\frac{8}{\sqrt{x-2}}}{h}$

$= \frac{8\left(\sqrt{x-2}-\sqrt{x+h-2}\right)}{h\sqrt{x+h-2}\sqrt{x-2}} \cdot \frac{\left(\sqrt{x-2}+\sqrt{x+h-2}\right)}{\left(\sqrt{x-2}+\sqrt{x+h-2}\right)} = \frac{8[(x-2)-(x+h-2)]}{h\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x+h-2}\right)}$

$= \frac{-8h}{h\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x+h-2}\right)} \Rightarrow f'(x) = \lim\limits_{h\to 0} \frac{-8}{\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x+h-2}\right)}$

$= \frac{-8}{\sqrt{x-2}\sqrt{x-2}\left(\sqrt{x-2}+\sqrt{x-2}\right)} = \frac{-4}{(x-2)\sqrt{x-2}}$; $m = f'(6) = \frac{-4}{4\sqrt{4}} = -\frac{1}{2} \Rightarrow$ the equation of the tangent line at $(6,4)$ is $y-4 = -\frac{1}{2}(x-6) \Rightarrow y = -\frac{1}{2}x+3+4 \Rightarrow y = -\frac{1}{2}x+7$.

18. $g'(z) = \lim\limits_{h\to 0} \frac{\left(1+\sqrt{4-(z+h)}\right)-\left(1+\sqrt{4-z}\right)}{h} = \lim\limits_{h\to 0} \frac{\left(\sqrt{4-z-h}-\sqrt{4-z}\right)}{h} \cdot \frac{\left(\sqrt{4-z-h}+\sqrt{4-z}\right)}{\left(\sqrt{4-z-h}+\sqrt{4-z}\right)}$

$= \lim\limits_{h\to 0} \frac{(4-z-h)-(4-z)}{h\left(\sqrt{4-z-h}+\sqrt{4-z}\right)} = \lim\limits_{h\to 0} \frac{-h}{h\left(\sqrt{4-z-h}+\sqrt{4-z}\right)} = \lim\limits_{h\to 0} \frac{-1}{\left(\sqrt{4-z-h}+\sqrt{4-z}\right)} = \frac{-1}{2\sqrt{4-z}}$;

$m = g'(3) = \frac{-1}{2\sqrt{4-3}} = -\frac{1}{2} \Rightarrow$ the equation of the tangent line at $(3,2)$ is $w-2 = -\frac{1}{2}(z-3)$

$\Rightarrow w = -\frac{1}{2}z+\frac{3}{2}+2 \Rightarrow w = -\frac{1}{2}z+\frac{7}{2}$.

19. $s = f(t) = 1-3t^2$ and $f(t+h) = 1-3(t+h)^2 = 1-3t^2-6th-3h^2 \Rightarrow \frac{ds}{dt} = \lim\limits_{h\to 0} \frac{f(t+h)-f(t)}{h}$

$= \lim\limits_{h\to 0} \frac{(1-3t^2-6th-3h^2)-(1-3t^2)}{h} = \lim\limits_{h\to 0} (-6t-3h) = -6t \Rightarrow \left.\frac{ds}{dt}\right|_{t=-1} = 6$

20. $y = f(x) = 1-\frac{1}{x}$ and $f(x+h) = 1-\frac{1}{x+h} \Rightarrow \frac{dy}{dx} = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \frac{\left(1-\frac{1}{x+h}\right)-\left(1-\frac{1}{x}\right)}{h}$

$= \lim\limits_{h\to 0} \frac{\frac{1}{x}-\frac{1}{x+h}}{h} = \lim\limits_{h\to 0} \frac{h}{x(x+h)h} = \lim\limits_{h\to 0} \frac{1}{x(x+h)} = \frac{1}{x^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=\sqrt{3}} = \frac{1}{3}$

21. $r = f(\theta) = \frac{2}{\sqrt{4-\theta}}$ and $f(\theta+h) = \frac{2}{\sqrt{4-(\theta+h)}} \Rightarrow \frac{dr}{d\theta} = \lim\limits_{h\to 0} \frac{f(\theta+h)-f(\theta)}{h} = \lim\limits_{h\to 0} \frac{\frac{2}{\sqrt{4-\theta-h}}-\frac{2}{\sqrt{4-\theta}}}{h}$

$= \lim\limits_{h\to 0} \frac{2\sqrt{4-\theta}-2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} = \lim\limits_{h\to 0} \frac{2\sqrt{4-\theta}-2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} \cdot \frac{\left(2\sqrt{4-\theta}+2\sqrt{4-\theta-h}\right)}{\left(2\sqrt{4-\theta}+2\sqrt{4-\theta-h}\right)}$

$$= \lim_{h \to 0} \frac{4(4-\theta) - 4(4-\theta-h)}{2h\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta}+\sqrt{4-\theta-h}\right)} = \lim_{h \to 0} \frac{2}{\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta}+\sqrt{4-\theta-h}\right)}$$

$$= \frac{2}{(4-\theta)\left(2\sqrt{4-\theta}\right)} = \frac{1}{(4-\theta)\sqrt{4-\theta}} \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=0} = \frac{1}{8}$$

22. $w = f(z) = z + \sqrt{z}$ and $f(z+h) = (z+h) + \sqrt{z+h} \Rightarrow \frac{dw}{dz} = \lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$

$$= \lim_{h \to 0} \frac{\left(z+h+\sqrt{z+h}\right) - \left(z+\sqrt{z}\right)}{h} = \lim_{h \to 0} \frac{h+\sqrt{z+h}-\sqrt{z}}{h} = \lim_{h \to 0} \left[1 + \frac{\sqrt{z+h}-\sqrt{z}}{h} \cdot \frac{\left(\sqrt{z+h}+\sqrt{z}\right)}{\left(\sqrt{z+h}+\sqrt{z}\right)}\right]$$

$$= 1 + \lim_{h \to 0} \frac{(z+h)-z}{h\left(\sqrt{z+h}+\sqrt{z}\right)} = 1 + \lim_{h \to 0} \frac{1}{\sqrt{z+h}+\sqrt{z}} = 1 + \frac{1}{2\sqrt{z}} \Rightarrow \left.\frac{dw}{dz}\right|_{z=4} = \frac{5}{4}$$

23. $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x} = \lim_{z \to x} \frac{\frac{1}{z+2} - \frac{1}{x+2}}{z-x} = \lim_{z \to x} \frac{(x+2)-(z+2)}{(z-x)(z+2)(x+2)} = \lim_{z \to x} \frac{x-z}{(z-x)(z+2)(x+2)} = \lim_{z \to x} \frac{-1}{(z+2)(x+2)} = \frac{-1}{(x+2)^2}$

24. $f'(x) = \lim_{z \to x} \frac{f(z)-f(x)}{z-x} = \lim_{z \to x} \frac{\left(z^2-3z+4\right) - \left(x^2-3x+4\right)}{z-x} = \lim_{z \to x} \frac{z^2-3z-x^2+3x}{z-x} = \lim_{z \to x} \frac{z^2-x^2-3z+3x}{z-x}$

$$= \lim_{z \to x} \frac{(z-x)(z+x)-3(z-x)}{z-x} = \lim_{z \to x} \frac{(z-x)[(z+x)-3]}{z-x} = \lim_{z \to x} [(z+x)-3] = 2x-3$$

25. $g'(x) = \lim_{z \to x} \frac{g(z)-g(x)}{z-x} = \lim_{z \to x} \frac{\frac{z}{z-1} - \frac{x}{x-1}}{z-x} = \lim_{z \to x} \frac{z(x-1)-x(z-1)}{(z-x)(z-1)(x-1)} = \lim_{z \to x} \frac{-z+x}{(z-x)(z-1)(x-1)} = \lim_{z \to x} \frac{-1}{(z-1)(x-1)} = \frac{-1}{(x-1)^2}$

26. $g'(x) = \lim_{z \to x} \frac{g(z)-g(x)}{z-x} = \lim_{z \to x} \frac{(1+\sqrt{z})-(1+\sqrt{x})}{z-x} = \lim_{z \to x} \frac{\sqrt{z}-\sqrt{x}}{z-x} \cdot \frac{\sqrt{z}+\sqrt{x}}{\sqrt{z}+\sqrt{x}} = \lim_{z \to x} \frac{z-x}{(z-x)\left(\sqrt{z}+\sqrt{x}\right)} = \lim_{z \to x} \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

27. Note that as x increases, the slope of the tangent line to the curve is first negative, then zero (when $x = 0$), then positive $\Rightarrow$ the slope is always increasing which matches (b).

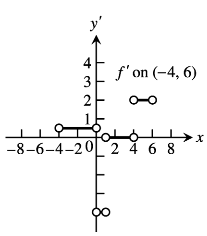
28. Note that the slope of the tangent line is never negative. For x negative, $f_2'(x)$ is positive but decreasing as x increases. When $x = 0$, the slope of the tangent line to x is 0. For $x > 0$, $f_2'(x)$ is positive and increasing. This graph matches (a).

29. $f_3(x)$ is an oscillating function like the cosine. Everywhere that the graph of f_3 has a horizontal tangent we expect f_3' to be zero, and (d) matches this condition.

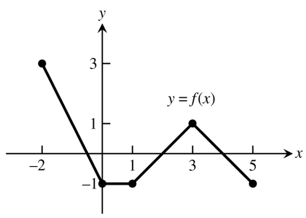
30. The graph matches with (c).

31. (a) f' is not defined at $x = 0, 1, 4$. At these points, the left-hand and right-hand derivatives do not agree. For example, $\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$ = slope of line joining $(-4, 0)$ and $(0, 2) = \frac{1}{2}$ but $\lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$ = slope of line joining $(0, 2)$ and $(1, -2) = -4$. Since these values are not equal, $f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$ does not exist.

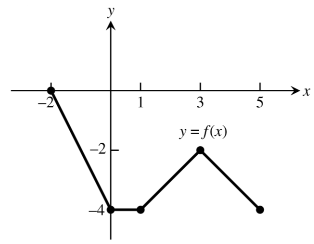
(b)



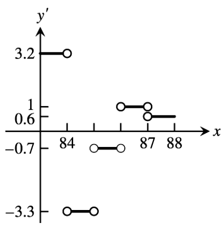
32. (a)



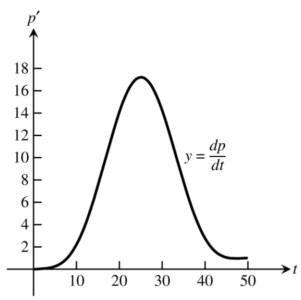
(b) Shift the graph in (a) down 3 units



33.



34. (a)



(b) The fastest is between the 20th and 30th days; slowest is between the 40th and 50th days.

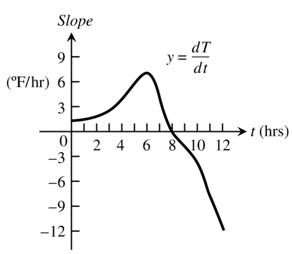
35. Answers may vary. In each case, draw a tangent line and estimate its slope.

(a) i) slope $\approx 1.54 \Rightarrow \frac{dT}{dt} \approx 1.54° \frac{F}{hr}$ ii) slope $\approx 2.86 \Rightarrow \frac{dT}{dt} \approx 2.86° \frac{F}{hr}$

iii) slope $\approx 0 \Rightarrow \frac{dT}{dt} \approx 0° \frac{F}{hr}$ iv) slope $\approx -3.75 \Rightarrow \frac{dT}{dt} \approx -3.75° \frac{F}{hr}$

(b) The tangent with the steepest positive slope appears to occur at $t = 6 \Rightarrow$ 12 p.m. and slope $\approx 7.27 \Rightarrow \frac{dT}{dt} \approx 7.27° \frac{F}{hr}$. The tangent with the steepest negative slope appears to occur at $t = 12 \Rightarrow$ 6 p.m. and slope $\approx -8.00 \Rightarrow \frac{dT}{dt} \approx -8.00° \frac{F}{hr}$

(c)



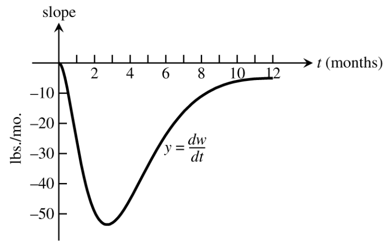
36. Answers may vary. In each case, draw a tangent line and estimate the slope.

(a) i) slope $\approx -20.83 \Rightarrow \frac{dW}{dt} \approx -20.83 \frac{lb}{month}$ ii) slope $\approx -35.00 \Rightarrow \frac{dW}{dt} \approx -35.00 \frac{lb}{month}$

iii) slope $\approx -6.25 \Rightarrow \frac{dW}{dt} \approx -6.25 \frac{lb}{month}$

(b) The tangentwith the steepest positive slope appears to occur at t = 2.7 months. and slope ≈ 7.27 $\Rightarrow \frac{dW}{dt} \approx -53.13 \frac{lb}{month}$

(c)



37. Left-hand derivative: For $h < 0$, $f(0+h) = f(h) = h^2$ (using $y = x^2$ curve) $\Rightarrow \lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$
$= \lim\limits_{h \to 0^-} \frac{h^2 - 0}{h} = \lim\limits_{h \to 0^-} h = 0;$
Right-hand derivative: For $h > 0$, $f(0+h) = f(h) = h$ (using $y = x$ curve) $\Rightarrow \lim\limits_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$
$= \lim\limits_{h \to 0^+} \frac{h - 0}{h} = \lim\limits_{h \to 0^+} 1 = 1;$
Then $\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \Rightarrow$ the derivative $f'(0)$ does not exist.

38. Left-hand derivative: When $h < 0$, $1 + h < 1 \Rightarrow f(1+h) = 2 \Rightarrow \lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0^-} \frac{2-2}{h}$
$= \lim\limits_{h \to 0^-} 0 = 0;$
Right-hand derivative: When $h > 0$, $1 + h > 1 \Rightarrow f(1+h) = 2(1+h) = 2 + 2h \Rightarrow \lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$
$= \lim\limits_{h \to 0^+} \frac{(2+2h)-2}{h} = \lim\limits_{h \to 0^+} \frac{2h}{h} = \lim\limits_{h \to 0^+} 2 = 2;$
Then $\lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

39. Left-hand derivative: When $h < 0$, $1 + h < 1 \Rightarrow f(1+h) = \sqrt{1+h} \Rightarrow \lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$
$= \lim\limits_{h \to 0^-} \frac{\sqrt{1+h} - 1}{h} = \lim\limits_{h \to 0^-} \frac{\left(\sqrt{1+h} - 1\right)}{h} \cdot \frac{\left(\sqrt{1+h} + 1\right)}{\left(\sqrt{1+h} + 1\right)} = \lim\limits_{h \to 0^-} \frac{(1+h) - 1}{h\left(\sqrt{1+h} + 1\right)} = \lim\limits_{h \to 0^-} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2};$
Right-hand derivative: When $h > 0$, $1 + h > 1 \Rightarrow f(1+h) = 2(1+h) - 1 = 2h + 1 \Rightarrow \lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$
$= \lim\limits_{h \to 0^+} \frac{(2h+1) - 1}{h} = \lim\limits_{h \to 0^+} 2 = 2;$
Then $\lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

40. Left-hand derivative: $\lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0^-} \frac{(1+h) - 1}{h} = \lim\limits_{h \to 0^-} 1 = 1;$
Right-hand derivative: $\lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0^+} \frac{\left(\frac{1}{1+h} - 1\right)}{h} = \lim\limits_{h \to 0^+} \frac{\left(\frac{1 - (1+h)}{1+h}\right)}{h}$
$= \lim\limits_{h \to 0^+} \frac{-h}{h(1+h)} = \lim\limits_{h \to 0^+} \frac{-1}{1+h} = -1;$
Then $\lim\limits_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

41. f is not continuous at $x = 0$ since $\lim\limits_{x \to 0} f(x) =$ does not exist and $f(0) = -1$

42. Left-hand derivative: $\lim\limits_{h \to 0^-} \frac{g(h) - g(0)}{h} = \lim\limits_{h \to 0^-} \frac{h^{1/3} - 0}{h} = \lim\limits_{h \to 0^-} \frac{1}{h^{2/3}} = +\infty;$
Right-hand derivative: $\lim\limits_{h \to 0^+} \frac{g(h) - g(0)}{h} = \lim\limits_{h \to 0^+} \frac{h^{2/3} - 0}{h} = \lim\limits_{h \to 0^+} \frac{1}{h^{1/3}} = +\infty;$
Then $\lim\limits_{h \to 0^-} \frac{g(h) - g(0)}{h} = \lim\limits_{h \to 0^+} \frac{g(h) - g(0)}{h} = +\infty \Rightarrow$ the derivative $g'(0)$ does not exist.

43. (a) The function is differentiable on its domain $-3 \le x \le 2$ (it is smooth)
(b) none
(c) none

44. (a) The function is differentiable on its domain $-2 \le x \le 3$ (it is smooth)
(b) none
(c) none

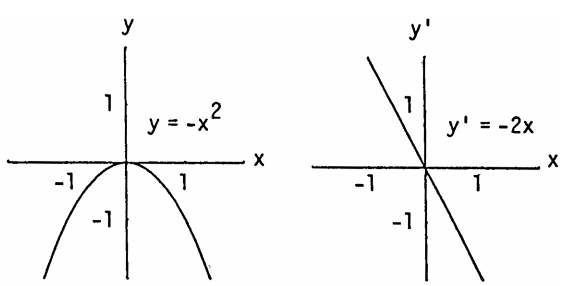
45. (a) The function is differentiable on $-3 \le x < 0$ and $0 < x \le 3$
(b) none
(c) The function is neither continuous nor differentiable at $x = 0$ since $\lim_{x \to 0^-} f(x) \ne \lim_{x \to 0^+} f(x)$

46. (a) f is differentiable on $-2 \le x < -1$, $-1 < x < 0$, $0 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -1$: $\lim_{x \to -1} f(x) = 0$ exists but there is a corner at $x = -1$ since $\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = -3$ and $\lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = 3 \Rightarrow f'(-1)$ does not exist
(c) f is neither continuous nor differentiable at $x = 0$ and $x = 2$:
at $x = 0$, $\lim_{x \to 0^-} f(x) = 3$ but $\lim_{x \to 0^+} f(x) = 0 \Rightarrow \lim_{x \to 0} f(x)$ does not exist;
at $x = 2$, $\lim_{x \to 2} f(x)$ exists but $\lim_{x \to 2} f(x) \ne f(2)$

47. (a) f is differentiable on $-1 \le x < 0$ and $0 < x \le 2$
(b) f is continuous but not differentiable at $x = 0$: $\lim_{x \to 0} f(x) = 0$ exists but there is a cusp at $x = 0$, so $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist
(c) none

48. (a) f is differentiable on $-3 \le x < -2$, $-2 < x < 2$, and $2 < x \le 3$
(b) f is continuous but not differentiable at $x = -2$ and $x = 2$: there are corners at those points
(c) none

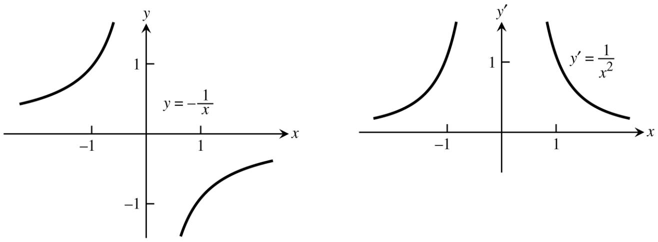
49. (a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x+h)^2 - (-x^2)}{h} = \lim_{h \to 0} \frac{-x^2 - 2xh - h^2 + x^2}{h} = \lim_{h \to 0} (-2x - h) = -2x$
(b)



(c) $y' = -2x$ is positive for $x < 0$, y' is zero when $x = 0$, y' is negative when $x > 0$
(d) $y = -x^2$ is increasing for $-\infty < x < 0$ and decreasing for $0 < x < \infty$; the function is increasing on intervals where $y' > 0$ and decreasing on intervals where $y' < 0$

50. (a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{-1}{x+h} - \frac{-1}{x}\right)}{h} = \lim_{h \to 0} \frac{-x + (x+h)}{x(x+h)h} = \lim_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}$

(b)

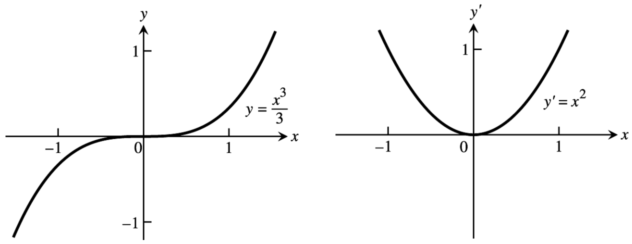


(c) y' is positive for all $x \neq 0$, y' is never 0, y' is never negative

(d) $y = -\frac{1}{x}$ is increasing for $-\infty < x < 0$ and $0 < x < \infty$

51. (a) Using the alternate formula for calculating derivatives: $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\left(\frac{z^3}{3} - \frac{x^3}{3}\right)}{z - x}$
$= \lim\limits_{z \to x} \frac{z^3 - x^3}{3(z - x)} = \lim\limits_{z \to x} \frac{(z - x)(z^2 + zx + x^2)}{3(z - x)} = \lim\limits_{z \to x} \frac{z^2 + zx + x^2}{3} = x^2 \Rightarrow f'(x) = x^2$

(b)

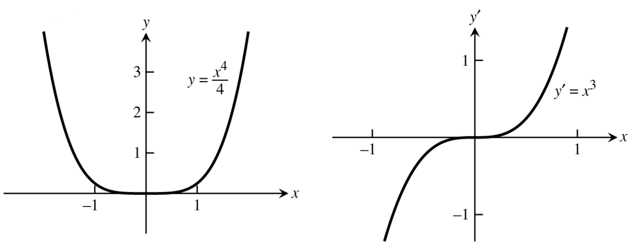


(c) y' is positive for all $x \neq 0$, and $y' = 0$ when $x = 0$; y' is never negative

(d) $y = \frac{x^3}{3}$ is increasing for all $x \neq 0$ (the graph is horizontal at $x = 0$) because y is increasing where $y' > 0$; y is never decreasing

52. (a) Using the alternate form for calculating derivatives: $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\left(\frac{z^4}{4} - \frac{x^4}{4}\right)}{z - x}$
$= \lim\limits_{z \to x} \frac{z^4 - x^4}{4(z - x)} = \lim\limits_{z \to x} \frac{(z - x)(z^3 + xz^2 + x^2z + x^3)}{4(z - x)} = \lim\limits_{z \to x} \frac{z^3 + xz^2 + x^2z + x^3}{4} = x^3 \Rightarrow f'(x) = x^3$

(b)



(c) y' is positive for $x > 0$, y' is zero for $x = 0$, y' is negative for $x < 0$

(d) $y = \frac{x^4}{4}$ is increasing on $0 < x < \infty$ and decreasing on $-\infty < x < 0$

53. $y' = \lim\limits_{h \to 0} \frac{(2(x+h)^2 - 13(x+h) + 5) - (2x^2 - 13x + 5)}{h} = \lim\limits_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$
$= \lim\limits_{h \to 0} \frac{4xh + 2h^2 - 13h}{h} = \lim\limits_{h \to 0} (4x + 2h - 13) = 4x - 13$, slope at x. The slope is -1 when $4x - 13 = -1$
$\Rightarrow 4x = 12 \Rightarrow x = 3 \Rightarrow y = 2 \cdot 3^2 - 13 \cdot 3 + 5 = -16$. Thus the tangent line is $y + 16 = (-1)(x - 3)$
$\Rightarrow y = -x - 13$ and the point of tangency is $(3, -16)$.

54. For the curve $y = \sqrt{x}$, we have $y' = \lim\limits_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim\limits_{h \to 0} \frac{(x+h) - x}{\left(\sqrt{x+h} + \sqrt{x}\right)h}$
$= \lim\limits_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. Suppose $\left(a, \sqrt{a}\right)$ is the point of tangency of such a line and $(-1, 0)$ is the point on the line where it crosses the x-axis. Then the slope of the line is $\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a+1}$ which must also equal

$\frac{1}{2\sqrt{a}}$; using the derivative formula at $x = a \Rightarrow \frac{\sqrt{a}}{a+1} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a + 1 \Rightarrow a = 1$. Thus such a line does exist: its point of tangency is $(1, 1)$, its slope is $\frac{1}{2\sqrt{a}} = \frac{1}{2}$; and an equation of the line is $y - 1 = \frac{1}{2}(x - 1)$ $\Rightarrow y = \frac{1}{2}x + \frac{1}{2}$.

55. Yes; the derivative of $-f$ is $-f'$ so that $f'(x_0)$ exists $\Rightarrow$ $-f'(x_0)$ exists as well.

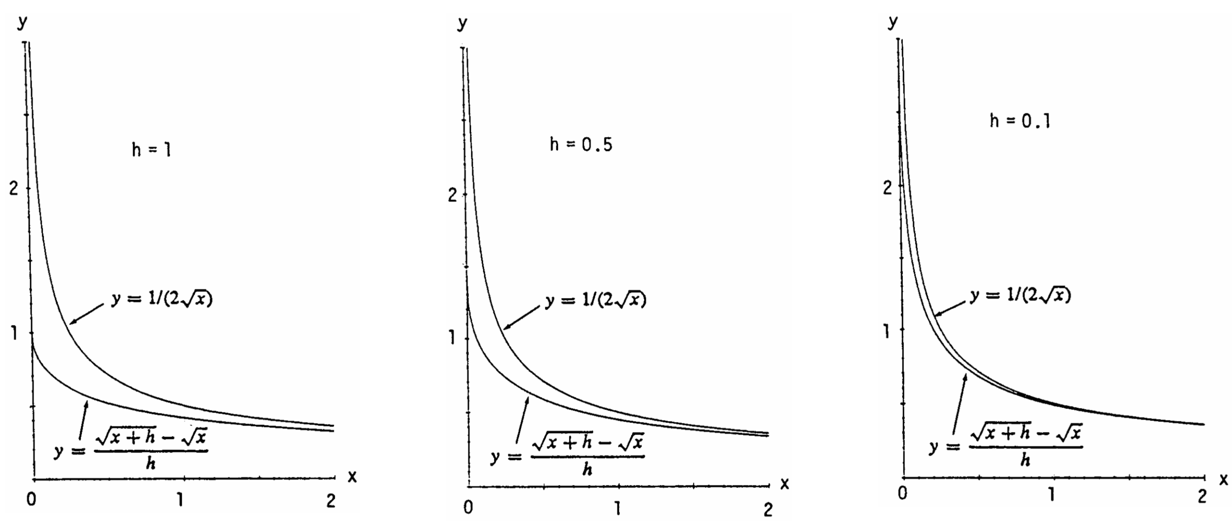
56. Yes; the derivative of $3g$ is $3g'$ so that $g'(7)$ exists $\Rightarrow$ $3g'(7)$ exists as well.

57. Yes, $\lim\limits_{t \to 0} \frac{g(t)}{h(t)}$ can exist but it need not equal zero. For example, let $g(t) = mt$ and $h(t) = t$. Then $g(0) = h(0) = 0$, but $\lim\limits_{t \to 0} \frac{g(t)}{h(t)} = \lim\limits_{t \to 0} \frac{mt}{t} = \lim\limits_{t \to 0} m = m$, which need not be zero.

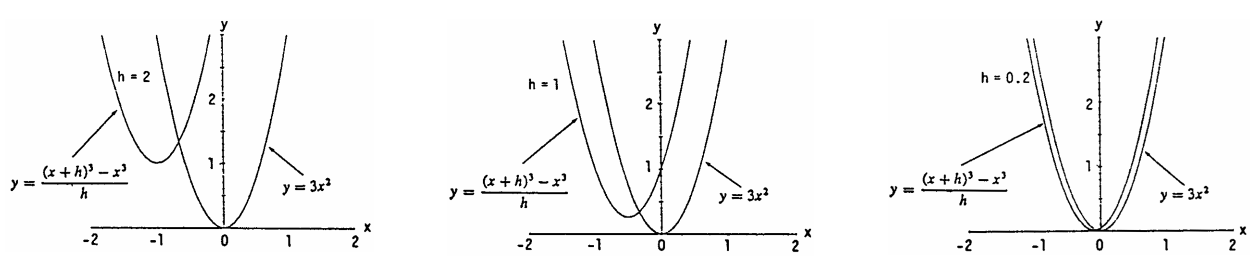
58. (a) Suppose $|f(x)| \le x^2$ for $-1 \le x \le 1$. Then $|f(0)| \le 0^2 \Rightarrow f(0) = 0$. Then $f'(0) = \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h}$ $= \lim\limits_{h \to 0} \frac{f(h) - 0}{h} = \lim\limits_{h \to 0} \frac{f(h)}{h}$. For $|h| \le 1$, $-h^2 \le f(h) \le h^2 \Rightarrow -h \le \frac{f(h)}{h} \le h \Rightarrow f'(0) = \lim\limits_{h \to 0} \frac{f(h)}{h} = 0$ by the Sandwich Theorem for limits.

(b) Note that for $x \ne 0$, $|f(x)| = \left|x^2 \sin \frac{1}{x}\right| = |x^2|\,|\sin x| \le |x^2| \cdot 1 = x^2$ (since $-1 \le \sin x \le 1$). By part (a), f is differentiable at $x = 0$ and $f'(0) = 0$.

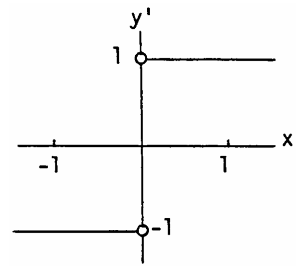
59. The graphs are shown below for $h = 1, 0.5, 0.1$. The function $y = \frac{1}{2\sqrt{x}}$ is the derivative of the function $y = \sqrt{x}$ so that $\frac{1}{2\sqrt{x}} = \lim\limits_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$. The graphs reveal that $y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$ gets closer to $y = \frac{1}{2\sqrt{x}}$ as h gets smaller and smaller.



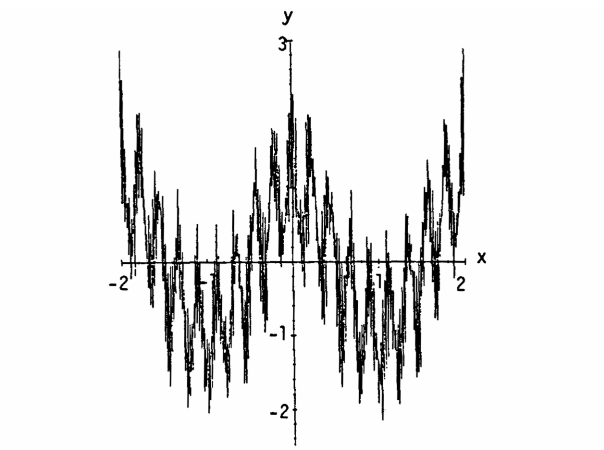
60. The graphs are shown below for $h = 2, 1, 0.5$. The function $y = 3x^2$ is the derivative of the function $y = x^3$ so that $3x^2 = \lim\limits_{h \to 0} \frac{(x+h)^3 - x^3}{h}$. The graphs reveal that $y = \frac{(x+h)^3 - x^3}{h}$ gets closer to $y = 3x^2$ as h gets smaller and smaller.



61. The graphs are the same. So we know that for $f(x) = |x|$, we have $f'(x) = \frac{|x|}{x}$.



62. Weierstrass's nowhere differentiable continuous function.



$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2\pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3\pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7\pi x)$$

63-68. Example CAS commands:

Maple:

```
f := x -> x^3 + x^2 - x;
x0 := 1;
plot( f(x), x=x0-5..x0+2, color=black,
        title="Section 3.2, #63(a)" );
q := unapply( (f(x+h)-f(x))/h, (x,h) );                        # (b)
L := limit( q(x,h), h=0 );                                      # (c)
m := eval( L, x=x0 );
tan_line := f(x0) + m*(x-x0);
plot( [f(x),tan_line], x=x0-2..x0+3, color=black,
       linestyle=[1,7], title="Section 3.2 #63(d)",
       legend=["y=f(x)","Tangent line at x=1"] );
Xvals := sort( [ x0+2^(-k) $ k=0..5, x0-2^(-k) $ k=0..5 ] ):     # (e)
Yvals := map( f, Xvals ):
evalf[4](< convert(Xvals,Matrix) , convert(Yvals,Matrix) >);
plot( L, x=x0-5..x0+3, color=black, title="Section 3.2 #63(f)" );
```

Mathematica: (functions and x0 may vary) (see section 2.5 re. RealOnly):

```
<<Miscellaneous`RealOnly`
Clear[f, m, x, y, h]
x0= π /4;
f[x_]:=x^2 Cos[x]
Plot[f[x], {x, x0 − 3, x0 + 3}]
```

```
q[x_, h_]:=(f[x + h] − f[x])/h
m[x_]:=Limit[q[x, h], h → 0]
ytan:=f[x0] + m[x0] (x − x0)
Plot[{f[x], ytan},{x, x0 − 3, x0 + 3}]
m[x0 − 1]//N
m[x0 + 1]//N
Plot[{f[x], m[x]},{x, x0 − 3, x0 + 3}]
```

3.3 DIFFERENTIATION RULES

1. $y = -x^2 + 3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x \Rightarrow \frac{d^2y}{dx^2} = -2$

2. $y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$

3. $s = 5t^3 - 3t^5 \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(5t^3) - \frac{d}{dt}(3t^5) = 15t^2 - 15t^4 \Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}(15t^2) - \frac{d}{dt}(15t^4) = 30t - 60t^3$

4. $w = 3z^7 - 7z^3 + 21z^2 \Rightarrow \frac{dw}{dz} = 21z^6 - 21z^2 + 42z \Rightarrow \frac{d^2w}{dz^2} = 126z^5 - 42z + 42$

5. $y = \frac{4}{3}x^3 - x \Rightarrow \frac{dy}{dx} = 4x^2 - 1 \Rightarrow \frac{d^2y}{dx^2} = 8x$

6. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4} \Rightarrow \frac{dy}{dx} = x^2 + x + \frac{1}{4} \Rightarrow \frac{d^2y}{dx^2} = 2x + 1 + 0 = 2x + 1$

7. $w = 3z^{-2} - z^{-1} \Rightarrow \frac{dw}{dz} = -6z^{-3} + z^{-2} = \frac{-6}{z^3} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 18z^{-4} - 2z^{-3} = \frac{18}{z^4} - \frac{2}{z^3}$

8. $s = -2t^{-1} + 4t^{-2} \Rightarrow \frac{ds}{dt} = 2t^{-2} - 8t^{-3} = \frac{2}{t^2} - \frac{8}{t^3} \Rightarrow \frac{d^2s}{dt^2} = -4t^{-3} + 24t^{-4} = \frac{-4}{t^3} + \frac{24}{t^4}$

9. $y = 6x^2 - 10x - 5x^{-2} \Rightarrow \frac{dy}{dx} = 12x - 10 + 10x^{-3} = 12x - 10 + \frac{10}{x^3} \Rightarrow \frac{d^2y}{dx^2} = 12 - 0 - 30x^{-4} = 12 - \frac{30}{x^4}$

10. $y = 4 - 2x - x^{-3} \Rightarrow \frac{dy}{dx} = -2 + 3x^{-4} = -2 + \frac{3}{x^4} \Rightarrow \frac{d^2y}{dx^2} = 0 - 12x^{-5} = \frac{-12}{x^5}$

11. $r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1} \Rightarrow \frac{dr}{ds} = -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2} \Rightarrow \frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$

12. $r = 12\theta^{-1} - 4\theta^{-3} + \theta^{-4} \Rightarrow \frac{dr}{d\theta} = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5} = \frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5} \Rightarrow \frac{d^2r}{d\theta^2} = 24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$
$= \frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$

13. (a) $y = (3 - x^2)(x^3 - x + 1) \Rightarrow y' = (3 - x^2) \cdot \frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1) \cdot \frac{d}{dx}(3 - x^2)$
$= (3 - x^2)(3x^2 - 1) + (x^3 - x + 1)(-2x) = -5x^4 + 12x^2 - 2x - 3$
(b) $y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$

14. (a) $y = (2x + 3)(5x^2 - 4x) \Rightarrow y' = (2x + 3)(10x - 4) + (5x^2 - 4x)(2) = 30x^2 + 14x - 12$
(b) $y = (2x + 3)(5x^2 - 4x) = 10x^3 + 7x^2 - 12x \Rightarrow y' = 30x^2 + 14x - 12$

15. (a) $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right) \Rightarrow y' = (x^2 + 1) \cdot \frac{d}{dx}\left(x + 5 + \frac{1}{x}\right) + \left(x + 5 + \frac{1}{x}\right) \cdot \frac{d}{dx}(x^2 + 1)$
$= (x^2 + 1)(1 - x^{-2}) + (x + 5 + x^{-1})(2x) = (x^2 - 1 + 1 - x^{-2}) + (2x^2 + 10x + 2) = 3x^2 + 10x + 2 - \frac{1}{x^2}$
(b) $y = x^3 + 5x^2 + 2x + 5 + \frac{1}{x} \Rightarrow y' = 3x^2 + 10x + 2 - \frac{1}{x^2}$

16. $y = (1+x^2)\left(x^{3/4} - x^{-3}\right)$

(a) $y' = (1+x^2)\cdot\left(\frac{3}{4}x^{-1/4} + 3x^{-4}\right) + \left(x^{3/4} - x^{-3}\right)(2x) = \frac{3}{4x^{1/4}} + \frac{3}{x^4} + \frac{11}{4}x^{7/4} + \frac{1}{x^2}$

(b) $y = x^{3/4} - x^{-3} + x^{11/4} - x^{-1} \Rightarrow y' = \frac{3}{4x^{1/4}} + \frac{3}{x^4} + \frac{11}{4}x^{7/4} + \frac{1}{x^2}$

17. $y = \frac{2x+5}{3x-2}$; use the quotient rule: $u = 2x+5$ and $v = 3x-2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu'-uv'}{v^2}$
$= \frac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2} = \frac{6x-4-6x-15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$

18. $y = \frac{4-3x}{3x^2+x}$;use the quotient rule: $u = 4-3x$ and $v = 3x^2+x \Rightarrow u' = -3$ and $v' = 6x+1 \Rightarrow y' = \frac{vu'-uv'}{v^2}$
$= \frac{(3x^2+x)(-3)-(4-3x)(6x+1)}{(3x^2+x)^2} = \frac{-9x^2-3x+18x^2-21x-4}{(3x^2+x)^2} = \frac{9x^2-24x-4}{(3x^2+x)^2}$

19. $g(x) = \frac{x^2-4}{x+0.5}$; use the quotient rule: $u = x^2-4$ and $v = x+0.5 \Rightarrow u' = 2x$ and $v' = 1 \Rightarrow g'(x) = \frac{vu'-uv'}{v^2}$
$= \frac{(x+0.5)(2x)-(x^2-4)(1)}{(x+0.5)^2} = \frac{2x^2+x-x^2+4}{(x+0.5)^2} = \frac{x^2+x+4}{(x+0.5)^2}$

20. $f(t) = \frac{t^2-1}{t^2+t-2} = \frac{(t-1)(t+1)}{(t+2)(t-1)} = \frac{t+1}{t+2}, t \neq 1 \Rightarrow f'(t) = \frac{(t+2)(1)-(t+1)(1)}{(t+2)^2} = \frac{t+2-t-1}{(t+2)^2} = \frac{1}{(t+2)^2}$

21. $v = (1-t)\left(1+t^2\right)^{-1} = \frac{1-t}{1+t^2} \Rightarrow \frac{dv}{dt} = \frac{(1+t^2)(-1)-(1-t)(2t)}{(1+t^2)^2} = \frac{-1-t^2-2t+2t^2}{(1+t^2)^2} = \frac{t^2-2t-1}{(1+t^2)^2}$

22. $w = \frac{x+5}{2x-7} \Rightarrow w' = \frac{(2x-7)(1)-(x+5)(2)}{(2x-7)^2} = \frac{2x-7-2x-10}{(2x-7)^2} = \frac{-17}{(2x-7)^2}$

23. $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1} \Rightarrow f'(s) = \frac{(\sqrt{s}+1)\left(\frac{1}{2\sqrt{s}}\right)-(\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}+1)^2} = \frac{(\sqrt{s}+1)-(\sqrt{s}-1)}{2\sqrt{s}\left(\sqrt{s}+1\right)^2} = \frac{1}{\sqrt{s}\left(\sqrt{s}+1\right)^2}$

NOTE: $\frac{d}{ds}\left(\sqrt{s}\right) = \frac{1}{2\sqrt{s}}$ from Example 2 in Section 3.2

24. $u = \frac{5x+1}{2\sqrt{x}} \Rightarrow \frac{du}{dx} = \frac{(2\sqrt{x})(5)-(5x+1)\left(\frac{1}{\sqrt{x}}\right)}{4x} = \frac{5x-1}{4x^{3/2}}$

25. $v = \frac{1+x-4\sqrt{x}}{x} \Rightarrow v' = \frac{x\left(1-\frac{2}{\sqrt{x}}\right)-(1+x-4\sqrt{x})}{x^2} = \frac{2\sqrt{x}-1}{x^2}$

26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right) \Rightarrow r' = 2\left(\frac{\sqrt{\theta}(0)-1\left(\frac{1}{2\sqrt{\theta}}\right)}{\theta} + \frac{1}{2\sqrt{\theta}}\right) = -\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}$

27. $y = \frac{1}{(x^2-1)(x^2+x+1)}$; use the quotient rule: $u = 1$ and $v = (x^2-1)(x^2+x+1) \Rightarrow u' = 0$ and
$v' = (x^2-1)(2x+1)+(x^2+x+1)(2x) = 2x^3+x^2-2x-1+2x^3+2x^2+2x = 4x^3+3x^2-1$
$\Rightarrow \frac{dy}{dx} = \frac{vu'-uv'}{v^2} = \frac{0-1(4x^3+3x^2-1)}{(x^2-1)^2(x^2+x+1)^2} = \frac{-4x^3-3x^2+1}{(x^2-1)^2(x^2+x+1)^2}$

28. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)} = \frac{x^2+3x+2}{x^2-3x+2} \Rightarrow y' = \frac{(x^2-3x+2)(2x+3)-(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2} = \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

29. $y = \frac{1}{2}x^4 - \frac{3}{2}x^2 - x \Rightarrow y' = 2x^3-3x-1 \Rightarrow y'' = 6x^2-3 \Rightarrow y''' = 12x \Rightarrow y^{(4)} = 12 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

30. $y = \frac{1}{120}x^5 \Rightarrow y' = \frac{1}{24}x^4 \Rightarrow y'' = \frac{1}{6}x^3 \Rightarrow y''' = \frac{1}{2}x^2 \Rightarrow y^{(4)} = x \Rightarrow y^{(5)} = 1 \Rightarrow y^{(n)} = 0$ for all $n \geq 6$

31. $y = (x-1)(x^2+3x-5) = x^3+2x^2-8x+5 \Rightarrow y' = 3x^2+4x-8 \Rightarrow y'' = 6x+4 \Rightarrow y''' = 6 \Rightarrow y^{(n)} = 0$ for all $n \geq 4$

32. $y = (4x^3 + 3x)(2 - x) = -4x^4 + 8x^3 - 3x^2 + 6x \Rightarrow y' = -16x^3 + 24x^2 - 6x + 6 \Rightarrow y'' = -48x^2 + 48x - 6$
$\Rightarrow y''' = -96x + 48 \Rightarrow y^{(4)} = -96 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

33. $y = \frac{x^3+7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} = 2x - \frac{7}{x^2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3} = 2 + \frac{14}{x^3}$

34. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3}$
$\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4}$

35. $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3} = \frac{\theta^3-1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^{-3} \Rightarrow \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} = \frac{3}{\theta^4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5} = \frac{-12}{\theta^5}$

36. $u = \frac{(x^2+x)(x^2-x+1)}{x^4} = \frac{x(x+1)(x^2-x+1)}{x^4} = \frac{x(x^3+1)}{x^4} = \frac{x^4+x}{x^4} = 1 + \frac{x}{x^4} = 1 + x^{-3}$
$\Rightarrow \frac{du}{dx} = 0 - 3x^{-4} = -3x^{-4} = \frac{-3}{x^4} \Rightarrow \frac{d^2u}{dx^2} = 12x^{-5} = \frac{12}{x^5}$

37. $w = \left(\frac{1+3z}{3z}\right)(3 - z) = \left(\frac{1}{3}z^{-1} + 1\right)(3 - z) = z^{-1} - \frac{1}{3} + 3 - z = z^{-1} + \frac{8}{3} - z \Rightarrow \frac{dw}{dz} = -z^{-2} + 0 - 1 = -z^{-2} - 1$
$= \frac{-1}{z^2} - 1 \Rightarrow \frac{d^2w}{dz^2} = 2z^{-3} - 0 = 2z^{-3} = \frac{2}{z^3}$

38. $w = (z+1)(z-1)(z^2+1) = (z^2-1)(z^2+1) = z^4 - 1 \Rightarrow \frac{dw}{dz} = 4z^3 - 0 = 4z^3 \Rightarrow \frac{d^2w}{dz^2} = 12z^2$

39. $p = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right) = \frac{q^6-q^2+3q^4-3}{12q^4} = \frac{1}{12}q^2 - \frac{1}{12}q^{-2} + \frac{1}{4} - \frac{1}{4}q^{-4} \Rightarrow \frac{dp}{dq} = \frac{1}{6}q + \frac{1}{6}q^{-3} + q^{-5} = \frac{1}{6}q + \frac{1}{6q^3} + \frac{1}{q^5}$
$\Rightarrow \frac{d^2p}{dq^2} = \frac{1}{6} - \frac{1}{2}q^{-4} - 5q^{-6} = \frac{1}{6} - \frac{1}{2q^4} - \frac{5}{q^6}$

40. $p = \frac{q^2+3}{(q-1)^3+(q+1)^3} = \frac{q^2+3}{(q^3-3q^2+3q-1)+(q^3+3q^2+3q+1)} = \frac{q^2+3}{2q^3+6q} = \frac{q^2+3}{2q(q^2+3)} = \frac{1}{2q} = \frac{1}{2}q^{-1}$
$\Rightarrow \frac{dp}{dq} = -\frac{1}{2}q^{-2} = -\frac{1}{2q^2} \Rightarrow \frac{d^2p}{dq^2} = q^{-3} = \frac{1}{q^3}$

41. $u(0) = 5,\ u'(0) = -3,\ v(0) = -1,\ v'(0) = 2$
(a) $\frac{d}{dx}(uv) = uv' + vu' \Rightarrow \frac{d}{dx}(uv)\Big|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(-3) = 13$
(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu'-uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=0} = \frac{v(0)u'(0)-u(0)v'(0)}{(v(0))^2} = \frac{(-1)(-3)-(5)(2)}{(-1)^2} = -7$
(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv'-vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=0} = \frac{u(0)v'(0)-v(0)u'(0)}{(u(0))^2} = \frac{(5)(2)-(-1)(-3)}{(5)^2} = \frac{7}{25}$
(d) $\frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \frac{d}{dx}(7v - 2u)\Big|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(-3) = 20$

42. $u(1) = 2,\ u'(1) = 0,\ v(1) = 5,\ v'(1) = -1$
(a) $\frac{d}{dx}(uv)\Big|_{x=1} = u(1)v'(1) + v(1)u'(1) = 2 \cdot (-1) + 5 \cdot 0 = -2$
(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=1} = \frac{v(1)u'(1)-u(1)v'(1)}{(v(1))^2} = \frac{5 \cdot 0 - 2 \cdot (-1)}{(5)^2} = \frac{2}{25}$
(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=1} = \frac{u(1)v'(1)-v(1)u'(1)}{(u(1))^2} = \frac{2 \cdot (-1) - 5 \cdot 0}{(2)^2} = -\frac{1}{2}$
(d) $\frac{d}{dx}(7v - 2u)\Big|_{x=1} = 7v'(1) - 2u'(1) = 7 \cdot (-1) - 2 \cdot 0 = -7$

43. $y = x^3 - 4x + 1$. Note that $(2, 1)$ is on the curve: $1 = 2^3 - 4(2) + 1$
(a) Slope of the tangent at (x, y) is $y' = 3x^2 - 4 \Rightarrow$ slope of the tangent at $(2, 1)$ is $y'(2) = 3(2)^2 - 4 = 8$. Thus the slope of the line perpendicular to the tangent at $(2, 1)$ is $-\frac{1}{8} \Rightarrow$ the equation of the line perpendicular to the tangent line at $(2, 1)$ is $y - 1 = -\frac{1}{8}(x - 2)$ or $y = -\frac{x}{8} + \frac{5}{4}$.
(b) The slope of the curve at x is $m = 3x^2 - 4$ and the smallest value for m is -4 when $x = 0$ and $y = 1$.

(c) We want the slope of the curve to be 8 $\Rightarrow$ $y' = 8 \Rightarrow 3x^2 - 4 = 8 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. When $x = 2$, $y = 1$ and the tangent line has equation $y - 1 = 8(x - 2)$ or $y = 8x - 15$; when $x = -2$, $y = (-2)^3 - 4(-2) + 1 = 1$, and the tangent line has equation $y - 1 = 8(x + 2)$ or $y = 8x + 17$.

44. (a) $y = x^3 - 3x - 2 \Rightarrow y' = 3x^2 - 3$. For the tangent to be horizontal, we need $m = y' = 0 \Rightarrow 0 = 3x^2 - 3 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$. When $x = -1$, $y = 0 \Rightarrow$ the tangent line has equation $y = 0$. The line perpendicular to this line at $(-1, 0)$ is $x = -1$. When $x = 1$, $y = -4$ $\Rightarrow$ the tangent line has equation $y = -4$. The line perpendicular to this line at $(1, -4)$ is $x = 1$.

(b) The smallest value of y' is -3, and this occurs when $x = 0$ and $y = -2$. The tangent to the curve at $(0, -2)$ has slope -3 $\Rightarrow$ the line perpendicular to the tangent at $(0, -2)$ has slope $\frac{1}{3}$ $\Rightarrow$ $y + 2 = \frac{1}{3}(x - 0)$ or $y = \frac{1}{3}x - 2$ is an equation of the perpendicular line.

45. $y = \frac{4x}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)(4)-(4x)(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4(-x^2+1)}{(x^2+1)^2}$. When $x = 0$, $y = 0$ and $y' = \frac{4(0+1)}{1} = 4$, so the tangent to the curve at $(0, 0)$ is the line $y = 4x$. When $x = 1$, $y = 2 \Rightarrow y' = 0$, so the tangent to the curve at $(1, 2)$ is the line $y = 2$.

46. $y = \frac{8}{x^2+4} \Rightarrow y' = \frac{(x^2+4)(0)-8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$. When $x = 2$, $y = 1$ and $y' = \frac{-16(2)}{(2^2+4)^2} = -\frac{1}{2}$, so the tangent line to the curve at $(2, 1)$ has the equation $y - 1 = -\frac{1}{2}(x - 2)$, or $y = -\frac{x}{2} + 2$.

47. $y = ax^2 + bx + c$ passes through $(0, 0)$ $\Rightarrow$ $0 = a(0) + b(0) + c$ $\Rightarrow$ $c = 0$; $y = ax^2 + bx$ passes through $(1, 2)$ $\Rightarrow$ $2 = a + b$; $y' = 2ax + b$ and since the curve is tangent to $y = x$ at the origin, its slope is 1 at $x = 0$ $\Rightarrow$ $y' = 1$ when $x = 0$ $\Rightarrow$ $1 = 2a(0) + b$ $\Rightarrow$ $b = 1$. Then $a + b = 2$ $\Rightarrow$ $a = 1$. In summary $a = b = 1$ and $c = 0$ so the curve is $y = x^2 + x$.

48. $y = cx - x^2$ passes through $(1, 0)$ $\Rightarrow$ $0 = c(1) - 1$ $\Rightarrow$ $c = 1$ $\Rightarrow$ the curve is $y = x - x^2$. For this curve, $y' = 1 - 2x$ and $x = 1$ $\Rightarrow$ $y' = -1$. Since $y = x - x^2$ and $y = x^2 + ax + b$ have common tangents at $x = 0$, $y = x^2 + ax + b$ must also have slope -1 at $x = 1$. Thus $y' = 2x + a$ $\Rightarrow$ $-1 = 2 \cdot 1 + a$ $\Rightarrow$ $a = -3$ $\Rightarrow$ $y = x^2 - 3x + b$. Since this last curve passes through $(1, 0)$, we have $0 = 1 - 3 + b$ $\Rightarrow$ $b = 2$. In summary, $a = -3$, $b = 2$ and $c = 1$ so the curves are $y = x^2 - 3x + 2$ and $y = x - x^2$.

49. $y = 8x + 5 \Rightarrow m = 8$; $f(x) = 3x^2 - 4x \Rightarrow f'(x) = 6x - 4$; $6x - 4 = 8 \Rightarrow x = 2 \Rightarrow f(2) = 3(2)^2 - 4(2) = 4 \Rightarrow (2, 4)$

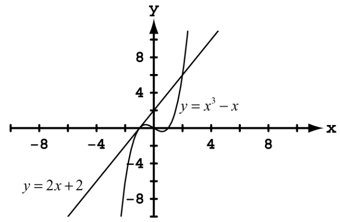
50. $8x - 2y = 1 \Rightarrow y = 4x - \frac{1}{2} \Rightarrow m = 4$; $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 1 \Rightarrow g'(x) = x^2 - 3x$; $x^2 - 3x = 4 \Rightarrow x = 4$ or $x = -1$ $\Rightarrow g(4) = \frac{1}{3}(4)^3 - \frac{3}{2}(4)^2 + 1 = -\frac{5}{3}$, $g(-1) = \frac{1}{3}(-1)^3 - \frac{3}{2}(-1)^2 + 1 = -\frac{5}{6} \Rightarrow \left(4, -\frac{5}{3}\right)$ or $\left(-1, -\frac{5}{6}\right)$

51. $y = 2x + 3 \Rightarrow m = 2 \Rightarrow m_{\perp} = -\frac{1}{2}$; $y = \frac{x}{x-2} \Rightarrow y' = \frac{(x-2)(1)-x(1)}{(x-2)^2} = \frac{-2}{(x-2)^2}$; $\frac{-2}{(x-2)^2} = -\frac{1}{2} \Rightarrow 4 = (x-2)^2$ $\Rightarrow \pm 2 = x - 2 \Rightarrow x = 4$ or $x = 0 \Rightarrow$ if $x = 4$, $y = \frac{4}{4-2} = 2$, and if $x = 0$, $y = \frac{0}{0-2} = 0 \Rightarrow (4, 2)$ or $(0, 0)$.

52. $m = \frac{y-8}{x-3}$; $f(x) = x^2 \Rightarrow f'(x) = 2x$; $m = f'(x) \Rightarrow \frac{y-8}{x-3} = 2x \Rightarrow \frac{x^2-8}{x-3} = 2x \Rightarrow x^2 - 8 = 2x^2 - 6x \Rightarrow x^2 - 6x + 8 = 0$ $\Rightarrow x = 4$ or $x = 2 \Rightarrow f(4) = 4^2 = 16$, $f(2) = 2^2 = 4 \Rightarrow (4, 16)$ or $(2, 4)$.

53. (a) $y = x^3 - x$ $\Rightarrow$ $y' = 3x^2 - 1$. When $x = -1$, $y = 0$ and $y' = 2$ $\Rightarrow$ the tangent line to the curve at $(-1, 0)$ is $y = 2(x + 1)$ or $y = 2x + 2$.

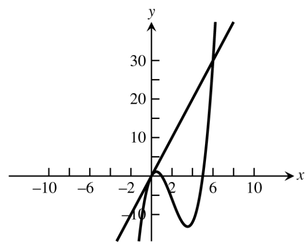
(b)



(c) $\left.\begin{array}{l} y = x^3 - x \\ y = 2x + 2 \end{array}\right\} \Rightarrow x^3 - x = 2x + 2 \Rightarrow x^3 - 3x - 2 = (x-2)(x+1)^2 = 0 \Rightarrow x = 2$ or $x = -1$. Since $y = 2(2) + 2 = 6$; the other intersection point is $(2, 6)$

54. (a) $y = x^3 - 6x^2 + 5x \Rightarrow y' = 3x^2 - 12x + 5$. When $x = 0$, $y = 0$ and $y' = 5 \Rightarrow$ the tangent line to the curve at $(0, 0)$ is $y = 5x$.

(b)



(c) $\left.\begin{array}{l} y = x^3 - 6x^2 + 5x \\ y = 5x \end{array}\right\} \Rightarrow x^3 - 6x^2 + 5x = 5x \Rightarrow x^3 - 6x^2 = 0 \Rightarrow x^2(x - 6) = 0 \Rightarrow x = 0$ or $x = 6$.
Since $y = 5(6) = 30$, the other intersection point is $(6, 30)$.

55. $\lim\limits_{x \to 1} \frac{x^{50} - 1}{x - 1} = 50\, x^{49}\Big|_{x=1} = 50\,(1)^{49} = 50$

56. $\lim\limits_{x \to -1} \frac{x^{2/9} - 1}{x + 1} = \frac{2}{9} x^{-7/9}\Big|_{x=-1} = \frac{2}{9(-1)^{7/9}} = -\frac{2}{9}$

57. $g'(x) = \begin{cases} 2x - 3 & x > 0 \\ a & x < 0 \end{cases}$, since g is differentiable at $x = 0 \Rightarrow \lim\limits_{x \to 0^+} (2x - 3) = -3$ and $\lim\limits_{x \to 0^-} a = a \Rightarrow a = -3$

58. $f'(x) = \begin{cases} a & x > -1 \\ 2bx & x < -1 \end{cases}$, since f is differentiable at $x = -1 \Rightarrow \lim\limits_{x \to -1^+} a = a$ and $\lim\limits_{x \to -1^-} (2bx) = -2b \Rightarrow a = -2b$, and since f is continuous at $x = -1 \Rightarrow \lim\limits_{x \to -1^+} (ax + b) = -a + b$ and $\lim\limits_{x \to -1^-} (bx^2 - 3) = b - 3 \Rightarrow -a + b = b - 3$ $\Rightarrow a = 3 \Rightarrow 3 = -2b \Rightarrow b = -\frac{3}{2}$.

59. $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \Rightarrow P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2a_2 x + a_1$

60. $R = M^2 \left(\frac{C}{2} - \frac{M}{3}\right) = \frac{C}{2} M^2 - \frac{1}{3} M^3$, where C is a constant $\Rightarrow \frac{dR}{dM} = CM - M^2$

61. Let c be a constant $\Rightarrow \frac{dc}{dx} = 0 \Rightarrow \frac{d}{dx}(u \cdot c) = u \cdot \frac{dc}{dx} + c \cdot \frac{du}{dx} = u \cdot 0 + c\, \frac{du}{dx} = c\, \frac{du}{dx}$. Thus when one of the functions is a constant, the Product Rule is just the Constant Multiple Rule $\Rightarrow$ the Constant Multiple Rule is a special case of the Product Rule.

62. (a) We use the Quotient rule to derive the Reciprocal Rule (with $u = 1$): $\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{v \cdot 0 - 1 \cdot \frac{dv}{dx}}{v^2} = \frac{-1 \cdot \frac{dv}{dx}}{v^2} = -\frac{1}{v^2} \cdot \frac{dv}{dx}$.

(b) Now, using the Reciprocal Rule and the Product Rule, we'll derive the Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}\left(u \cdot \frac{1}{v}\right)$
$= u \cdot \frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v} \cdot \frac{du}{dx}$ (Product Rule) $= u \cdot \left(\frac{-1}{v^2}\right)\frac{dv}{dx} + \frac{1}{v}\frac{du}{dx}$ (Reciprocal Rule) $\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{-u\frac{dv}{dx} + v\frac{du}{dx}}{v^2}$
$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, the Quotient Rule.

63. (a) $\frac{d}{dx}(uvw) = \frac{d}{dx}((uv) \cdot w) = (uv)\frac{dw}{dx} + w \cdot \frac{d}{dx}(uv) = uv\frac{dw}{dx} + w\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) = uv\frac{dw}{dx} + wu\frac{dv}{dx} + wv\frac{du}{dx}$
$= uvw' + uv'w + u'vw$
(b) $\frac{d}{dx}(u_1u_2u_3u_4) = \frac{d}{dx}((u_1u_2u_3)u_4) = (u_1u_2u_3)\frac{du_4}{dx} + u_4\frac{d}{dx}(u_1u_2u_3) \Rightarrow \frac{d}{dx}(u_1u_2u_3u_4)$
$= u_1u_2u_3\frac{du_4}{dx} + u_4\left(u_1u_2\frac{du_3}{dx} + u_3u_1\frac{du_2}{dx} + u_3u_2\frac{du_1}{dx}\right)$ (using (a) above)
$\Rightarrow \frac{d}{dx}(u_1u_2u_3u_4) = u_1u_2u_3\frac{du_4}{dx} + u_1u_2u_4\frac{du_3}{dx} + u_1u_3u_4\frac{du_2}{dx} + u_2u_3u_4\frac{du_1}{dx}$
$= u_1u_2u_3u_4' + u_1u_2u_3'u_4 + u_1u_2'u_3u_4 + u_1'u_2u_3u_4$
(c) Generalizing (a) and (b) above, $\frac{d}{dx}(u_1 \cdots u_n) = u_1u_2 \cdots u_{n-1}u_n' + u_1u_2 \cdots u_{n-2}u_{n-1}'u_n + \ldots + u_1'u_2 \cdots u_n$

64. $\frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \cdot 0 - 1(m \cdot x^{m-1})}{(x^m)^2} = \frac{-m \cdot x^{m-1}}{x^{2m}} = -m \cdot x^{m-1-2m} = -m \cdot x^{-m-1}$

65. $P = \frac{nRT}{V-nb} - \frac{an^2}{V^2}$. We are holding T constant, and a, b, n, R are also constant so their derivatives are zero
$\Rightarrow \frac{dP}{dV} = \frac{(V-nb) \cdot 0 - (nRT)(1)}{(V-nb)^2} - \frac{V^2(0) - (an^2)(2V)}{(V^2)^2} = \frac{-nRT}{(V-nb)^2} + \frac{2an^2}{V^3}$

66. $A(q) = \frac{km}{q} + cm + \frac{hq}{2} = (km)q^{-1} + cm + \left(\frac{h}{2}\right)q \Rightarrow \frac{dA}{dq} = -(km)q^{-2} + \left(\frac{h}{2}\right) = -\frac{km}{q^2} + \frac{h}{2} \Rightarrow \frac{d^2A}{dt^2} = 2(km)q^{-3} = \frac{2km}{q^3}$

3.4 THE DERIVATIVE AS A RATE OF CHANGE

1. $s = t^2 - 3t + 2,\ 0 \le t \le 2$
(a) displacement $= \Delta s = s(2) - s(0) = 0\text{m} - 2\text{m} = -2$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-2}{2} = -1$ m/sec
(b) $v = \frac{ds}{dt} = 2t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(2)| = 1$ m/sec;
$a = \frac{d^2s}{dt^2} = 2 \Rightarrow a(0) = 2$ m/sec^2 and $a(2) = 2$ m/sec^2
(c) $v = 0 \Rightarrow 2t - 3 = 0 \Rightarrow t = \frac{3}{2}$. v is negative in the interval $0 < t < \frac{3}{2}$ and v is positive when $\frac{3}{2} < t < 2 \Rightarrow$ the body changes direction at $t = \frac{3}{2}$.

2. $s = 6t - t^2,\ 0 \le t \le 6$
(a) displacement $= \Delta s = s(6) - s(0) = 0$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{0}{6} = 0$ m/sec
(b) $v = \frac{ds}{dt} = 6 - 2t \Rightarrow |v(0)| = |6| = 6$ m/sec and $|v(6)| = |-6| = 6$ m/sec;
$a = \frac{d^2s}{dt^2} = -2 \Rightarrow a(0) = -2$ m/sec^2 and $a(6) = -2$ m/sec^2
(c) $v = 0 \Rightarrow 6 - 2t = 0 \Rightarrow t = 3$. v is positive in the interval $0 < t < 3$ and v is negative when $3 < t < 6 \Rightarrow$ the body changes direction at $t = 3$.

3. $s = -t^3 + 3t^2 - 3t,\ 0 \le t \le 3$
(a) displacement $= \Delta s = s(3) - s(0) = -9$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-9}{3} = -3$ m/sec
(b) $v = \frac{ds}{dt} = -3t^2 + 6t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(3)| = |-12| = 12$ m/sec; $a = \frac{d^2s}{dt^2} = -6t + 6$
$\Rightarrow a(0) = 6$ m/sec^2 and $a(3) = -12$ m/sec^2
(c) $v = 0 \Rightarrow -3t^2 + 6t - 3 = 0 \Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0 \Rightarrow t = 1$. For all other values of t in the interval the velocity v is negative (the graph of $v = -3t^2 + 6t - 3$ is a parabola with vertex at $t = 1$ which opens downward $\Rightarrow$ the body never changes direction).

4. $s = \frac{t^4}{4} - t^3 + t^2,\ 0 \le t \le 3$
   (a) $\Delta s = s(3) - s(0) = \frac{9}{4}$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{\frac{9}{4}}{3} = \frac{3}{4}$ m/sec
   (b) $v = t^3 - 3t^2 + 2t \Rightarrow |v(0)| = 0$ m/sec and $|v(3)| = 6$ m/sec; $a = 3t^2 - 6t + 2 \Rightarrow a(0) = 2$ m/sec^2 and $a(3) = 11$ m/sec^2
   (c) $v = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-2)(t-1) = 0 \Rightarrow t = 0, 1, 2 \Rightarrow v = t(t-2)(t-1)$ is positive in the interval for $0 < t < 1$ and v is negative for $1 < t < 2$ and v is positive for $2 < t < 3 \Rightarrow$ the body changes direction at $t = 1$ and at $t = 2$.

5. $s = \frac{25}{t^2} - \frac{5}{t},\ 1 \le t \le 5$
   (a) $\Delta s = s(5) - s(1) = -20$ m, $v_{av} = \frac{-20}{4} = -5$ m/sec
   (b) $v = \frac{-50}{t^3} + \frac{5}{t^2} \Rightarrow |v(1)| = 45$ m/sec and $|v(5)| = \frac{1}{5}$ m/sec; $a = \frac{150}{t^4} - \frac{10}{t^3} \Rightarrow a(1) = 140$ m/sec^2 and $a(5) = \frac{4}{25}$ m/sec^2
   (c) $v = 0 \Rightarrow \frac{-50+5t}{t^3} = 0 \Rightarrow -50 + 5t = 0 \Rightarrow t = 10 \Rightarrow$ the body does not change direction in the interval

6. $s = \frac{25}{t+5},\ -4 \le t \le 0$
   (a) $\Delta s = s(0) - s(-4) = -20$ m, $v_{av} = -\frac{20}{4} = -5$ m/sec
   (b) $v = \frac{-25}{(t+5)^2} \Rightarrow |v(-4)| = 25$ m/sec and $|v(0)| = 1$ m/sec; $a = \frac{50}{(t+5)^3} \Rightarrow a(-4) = 50$ m/sec^2 and $a(0) = \frac{2}{5}$ m/sec^2
   (c) $v = 0 \Rightarrow \frac{-25}{(t+5)^2} = 0 \Rightarrow$ v is never $0 \Rightarrow$ the body never changes direction

7. $s = t^3 - 6t^2 + 9t$ and let the positive direction be to the right on the s-axis.
   (a) $v = 3t^2 - 12t + 9$ so that $v = 0 \Rightarrow t^2 - 4t + 3 = (t-3)(t-1) = 0 \Rightarrow t = 1$ or 3; $a = 6t - 12 \Rightarrow a(1) = -6$ m/sec^2 and $a(3) = 6$ m/sec^2. Thus the body is motionless but being accelerated left when $t = 1$, and motionless but being accelerated right when $t = 3$.
   (b) $a = 0 \Rightarrow 6t - 12 = 0 \Rightarrow t = 2$ with speed $|v(2)| = |12 - 24 + 9| = 3$ m/sec
   (c) The body moves to the right or forward on $0 \le t < 1$, and to the left or backward on $1 < t < 2$. The positions are $s(0) = 0$, $s(1) = 4$ and $s(2) = 2 \Rightarrow$ total distance $= |s(1) - s(0)| + |s(2) - s(1)| = |4| + |-2| = 6$ m.

8. $v = t^2 - 4t + 3 \Rightarrow a = 2t - 4$
   (a) $v = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $3 \Rightarrow a(1) = -2$ m/sec^2 and $a(3) = 2$ m/sec^2
   (b) $v > 0 \Rightarrow (t-3)(t-1) > 0 \Rightarrow 0 \le t < 1$ or $t > 3$ and the body is moving forward; $v < 0 \Rightarrow (t-3)(t-1) < 0 \Rightarrow 1 < t < 3$ and the body is moving backward
   (c) velocity increasing $\Rightarrow a > 0 \Rightarrow 2t - 4 > 0 \Rightarrow t > 2$; velocity decreasing $\Rightarrow a < 0 \Rightarrow 2t - 4 < 0 \Rightarrow 0 \le t < 2$

9. $s_m = 1.86t^2 \Rightarrow v_m = 3.72t$ and solving $3.72t = 27.8 \Rightarrow t \approx 7.5$ sec on Mars; $s_j = 11.44t^2 \Rightarrow v_j = 22.88t$ and solving $22.88t = 27.8 \Rightarrow t \approx 1.2$ sec on Jupiter.

10. (a) $v(t) = s'(t) = 24 - 1.6t$ m/sec, and $a(t) = v'(t) = s''(t) = -1.6$ m/sec^2
   (b) Solve $v(t) = 0 \Rightarrow 24 - 1.6t = 0 \Rightarrow t = 15$ sec
   (c) $s(15) = 24(15) - .8(15)^2 = 180$ m
   (d) Solve $s(t) = 90 \Rightarrow 24t - .8t^2 = 90 \Rightarrow t = \frac{30 \pm 15\sqrt{2}}{2} \approx 4.39$ sec going up and 25.6 sec going down
   (e) Twice the time it took to reach its highest point or 30 sec

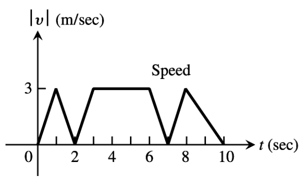
11. $s = 15t - \frac{1}{2} g_s t^2 \Rightarrow v = 15 - g_s t$ so that $v = 0 \Rightarrow 15 - g_s t = 0 \Rightarrow g_s = \frac{15}{t}$. Therefore $g_s = \frac{15}{20} = \frac{3}{4} = 0.75$ m/sec^2

12. Solving $s_m = 832t - 2.6t^2 = 0 \Rightarrow t(832 - 2.6t) = 0 \Rightarrow t = 0$ or $320 \Rightarrow$ 320 sec on the moon; solving $s_e = 832t - 16t^2 = 0 \Rightarrow t(832 - 16t) = 0 \Rightarrow t = 0$ or $52 \Rightarrow$ 52 sec on the earth. Also, $v_m = 832 - 5.2t = 0 \Rightarrow t = 160$ and $s_m(160) = 66{,}560$ ft, the height it reaches above the moon's surface; $v_e = 832 - 32t = 0 \Rightarrow t = 26$ and $s_e(26) = 10{,}816$ ft, the height it reaches above the earth's surface.

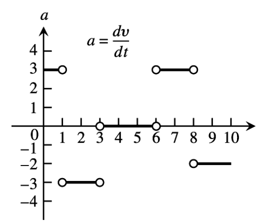
13. (a) $s = 179 - 16t^2 \Rightarrow v = -32t \Rightarrow$ speed $= |v| = 32t$ ft/sec and $a = -32$ ft/sec^2
    (b) $s = 0 \Rightarrow 179 - 16t^2 = 0 \Rightarrow t = \sqrt{\frac{179}{16}} \approx 3.3$ sec
    (c) When $t = \sqrt{\frac{179}{16}}$, $v = -32\sqrt{\frac{179}{16}} = -8\sqrt{179} \approx -107.0$ ft/sec

14. (a) $\lim_{\theta \to \frac{\pi}{2}} v = \lim_{\theta \to \frac{\pi}{2}} 9.8(\sin\theta)t = 9.8t$ so we expect $v = 9.8t$ m/sec in free fall
    (b) $a = \frac{dv}{dt} = 9.8$ m/sec^2

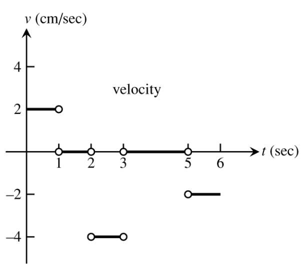
15. (a) at 2 and 7 seconds
    (b) between 3 and 6 seconds: $3 \le t \le 6$
    (c)

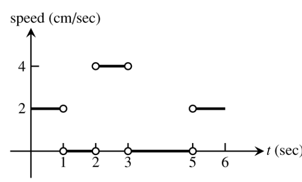


    (d)



16. (a) P is moving to the left when $2 < t < 3$ or $5 < t < 6$; P is moving to the right when $0 < t < 1$; P is standing still when $1 < t < 2$ or $3 < t < 5$
    (b)





17. (a) 190 ft/sec
    (b) 2 sec
    (c) at 8 sec, 0 ft/sec
    (d) 10.8 sec, 90 ft/sec
    (e) From $t = 8$ until $t = 10.8$ sec, a total of 2.8 sec
    (f) Greatest acceleration happens 2 sec after launch
    (g) From $t = 2$ to $t = 10.8$ sec; during this period, $a = \frac{v(10.8) - v(2)}{10.8 - 2} \approx -32$ ft/sec^2

18. (a) Forward: $0 \le t < 1$ and $5 < t < 7$; Backward: $1 < t < 5$; Speeds up: $1 < t < 2$ and $5 < t < 6$; Slows down: $0 \le t < 1$, $3 < t < 5$, and $6 < t < 7$
    (b) Positive: $3 < t < 6$; negative: $0 \le t < 2$ and $6 < t < 7$; zero: $2 < t < 3$ and $7 < t < 9$
    (c) $t = 0$ and $2 \le t \le 3$
    (d) $7 \le t \le 9$

19. $s = 490t^2 \Rightarrow v = 980t \Rightarrow a = 980$
    (a) Solving $160 = 490t^2 \Rightarrow t = \frac{4}{7}$ sec. The average velocity was $\frac{s(4/7) - s(0)}{4/7} = 280$ cm/sec.

(b) At the 160 cm mark the balls are falling at $v(4/7) = 560$ cm/sec. The acceleration at the 160 cm mark was $980\ \text{cm/sec}^2$.

(c) The light was flashing at a rate of $\frac{17}{4/7} = 29.75$ flashes per second.

20. (a)

(b)

21. C = position, A = velocity, and B = acceleration. Neither A nor C can be the derivative of B because B's derivative is constant. Graph C cannot be the derivative of A either, because A has some negative slopes while C has only positive values. So, C (being the derivative of neither A nor B) must be the graph of position. Curve C has both positive and negative slopes, so its derivative, the velocity, must be A and not B. That leaves B for acceleration.

22. C = position, B = velocity, and A = acceleration. Curve C cannot be the derivative of either A or B because C has only negative values while both A and B have some positive slopes. So, C represents position. Curve C has no positive slopes, so its derivative, the velocity, must be B. That leaves A for acceleration. Indeed, A is negative where B has negative slopes and positive where B has positive slopes.

23. (a) $c(100) = 11{,}000 \Rightarrow c_{av} = \frac{11{,}000}{100} = \$110$

(b) $c(x) = 2000 + 100x - .1x^2 \Rightarrow c'(x) = 100 - .2x$. Marginal cost $= c'(x) \Rightarrow$ the marginal cost of producing 100 machines is $c'(100) = \$80$

(c) The cost of producing the 101st machine is $c(101) - c(100) = 100 - \frac{201}{10} = \$79.90$

24. (a) $r(x) = 20000\left(1 - \frac{1}{x}\right) \Rightarrow r'(x) = \frac{20000}{x^2}$, which is marginal revenue. $r'(100) = \frac{20000}{100^2} = \$2$.

(b) $r'(101) = \$1.96$.

(c) $\lim_{x \to \infty} r'(x) = \lim_{x \to \infty} \frac{20000}{x^2} = 0$. The increase in revenue as the number of items increases without bound will approach zero.

25. $b(t) = 10^6 + 10^4 t - 10^3 t^2 \Rightarrow b'(t) = 10^4 - (2)(10^3 t) = 10^3(10 - 2t)$

(a) $b'(0) = 10^4$ bacteria/hr

(b) $b'(5) = 0$ bacteria/hr

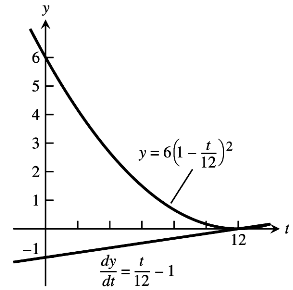
(c) $b'(10) = -10^4$ bacteria/hr

26. $Q(t) = 200(30 - t)^2 = 200\left(900 - 60t + t^2\right) \Rightarrow Q'(t) = 200(-60 + 2t) \Rightarrow Q'(10) = -8{,}000$ gallons/min is the rate the water is running at the end of 10 min. Then $\frac{Q(10) - Q(0)}{10} = -10{,}000$ gallons/min is the average rate the water flows during the first 10 min. The negative signs indicate water is <u>leaving</u> the tank.

27. (a) $y = 6\left(1 - \frac{t}{12}\right)^2 = 6\left(1 - \frac{t}{6} + \frac{t^2}{144}\right) \Rightarrow \frac{dy}{dt} = \frac{t}{12} - 1$

(b) The largest value of $\frac{dy}{dt}$ is 0 m/h when $t = 12$ and the fluid level is falling the slowest at that time. The smallest value of $\frac{dy}{dt}$ is -1 m/h, when $t = 0$, and the fluid level is falling the fastest at that time.

(c) In this situation, $\frac{dy}{dt} \le 0 \Rightarrow$ the graph of y is always decreasing. As $\frac{dy}{dt}$ increases in value, the slope of the graph of y increases from -1 to 0 over the interval $0 \le t \le 12$.



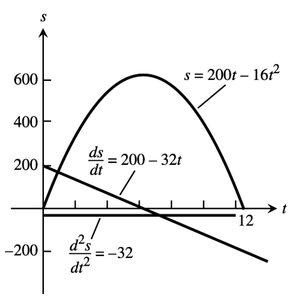
28. (a) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi \text{ ft}^3/\text{ft}$

(b) When $r = 2$, $\frac{dV}{dr} = 16\pi$ so that when r changes by 1 unit, we expect V to change by approximately 16π. Therefore when r changes by 0.2 units V changes by approximately $(16\pi)(0.2) = 3.2\pi \approx 10.05 \text{ ft}^3$. Note that $V(2.2) - V(2) \approx 11.09 \text{ ft}^3$.

29. $200 \text{ km/hr} = 55\frac{5}{9} \text{m/sec} = \frac{500}{9}$ m/sec, and $D = \frac{10}{9}t^2 \Rightarrow V = \frac{20}{9}t$. Thus $V = \frac{500}{9} \Rightarrow \frac{20}{9}t = \frac{500}{9} \Rightarrow t = 25$ sec. When $t = 25$, $D = \frac{10}{9}(25)^2 = \frac{6250}{9}$ m

30. $s = v_0 t - 16t^2 \Rightarrow v = v_0 - 32t$; $v = 0 \Rightarrow t = \frac{v_0}{32}$; $1900 = v_0 t - 16t^2$ so that $t = \frac{v_0}{32} \Rightarrow 1900 = \frac{v_0^2}{32} - \frac{v_0^2}{64}$ $\Rightarrow v_0 = \sqrt{(64)(1900)} = 80\sqrt{19}$ ft/sec and, finally, $\frac{80\sqrt{19}\text{ ft}}{\text{sec}} \cdot \frac{60 \text{ sec}}{1 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 238$ mph.

31.



(a) $v = 0$ when $t = 6.25$ sec

(b) $v > 0$ when $0 \le t < 6.25 \Rightarrow$ body moves right (up); $v < 0$ when $6.25 < t \le 12.5 \Rightarrow$ body moves left (down)

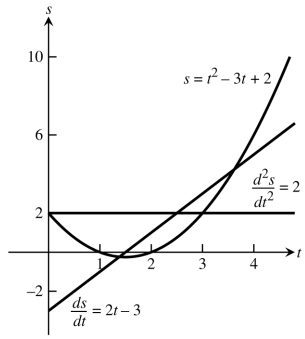
(c) body changes direction at $t = 6.25$ sec

(d) body speeds up on $(6.25, 12.5]$ and slows down on $[0, 6.25)$

(e) The body is moving fastest at the endpoints $t = 0$ and $t = 12.5$ when it is traveling 200 ft/sec. It's moving slowest at $t = 6.25$ when the speed is 0.

(f) When $t = 6.25$ the body is $s = 625$ m from the origin and farthest away.

32.



(a) v = 0 when t = $\frac{3}{2}$ sec

(b) v < 0 when 0 ≤ t < 1.5 ⇒ body moves left (down); v > 0 when 1.5 < t ≤ 5 ⇒ body moves right (up)

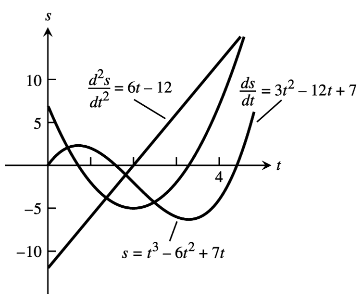
(c) body changes direction at t = $\frac{3}{2}$ sec

(d) body speeds up on $\left(\frac{3}{2}, 5\right]$ and slows down on $\left[0, \frac{3}{2}\right)$

(e) body is moving fastest at t = 5 when the speed = |v(5)| = 7 units/sec; it is moving slowest at t = $\frac{3}{2}$ when the speed is 0

(f) When t = 5 the body is s = 12 units from the origin and farthest away.

33.



(a) v = 0 when t = $\frac{6 \pm \sqrt{15}}{3}$ sec

(b) v < 0 when $\frac{6-\sqrt{15}}{3} < t < \frac{6+\sqrt{15}}{3}$ ⇒ body moves left (down); v > 0 when $0 \le t < \frac{6-\sqrt{15}}{3}$ or $\frac{6+\sqrt{15}}{3} < t \le 4$ ⇒ body moves right (up)

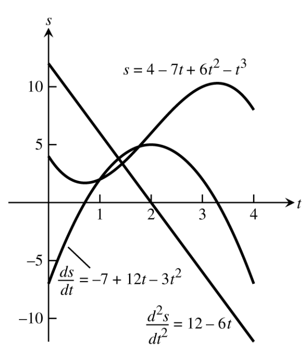
(c) body changes direction at t = $\frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6-\sqrt{15}}{3}, 2\right) \cup \left(\frac{6+\sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6-\sqrt{15}}{3}\right) \cup \left(2, \frac{6+\sqrt{15}}{3}\right)$.

(e) The body is moving fastest at t = 0 and t = 4 when it is moving 7 units/sec and slowest at t = $\frac{6\pm\sqrt{15}}{3}$ sec

(f) When t = $\frac{6+\sqrt{15}}{3}$ the body is at position s ≈ −6.303 units and farthest from the origin.

34.



(a) v = 0 when t = $\frac{6 \pm \sqrt{15}}{3}$

(b) $v < 0$ when $0 \le t < \frac{6-\sqrt{15}}{3}$ or $\frac{6+\sqrt{15}}{3} < t \le 4 \Rightarrow$ body is moving left (down); $v > 0$ when $\frac{6-\sqrt{15}}{3} < t < \frac{6+\sqrt{15}}{3} \Rightarrow$ body is moving right (up)

(c) body changes direction at $t = \frac{6\pm\sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6-\sqrt{15}}{3}, 2\right) \cup \left(\frac{6+\sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6-\sqrt{15}}{3}\right) \cup \left(2, \frac{6+\sqrt{15}}{3}\right)$

(e) The body is moving fastest at 7 units/sec when $t = 0$ and $t = 4$; it is moving slowest and stationary at $t = \frac{6\pm\sqrt{15}}{3}$

(f) When $t = \frac{6+\sqrt{15}}{3}$ the position is $s \approx 10.303$ units and the body is farthest from the origin.

3.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. $y = -10x + 3\cos x \Rightarrow \frac{dy}{dx} = -10 + 3\frac{d}{dx}(\cos x) = -10 - 3\sin x$

2. $y = \frac{3}{x} + 5\sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5\frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5\cos x$

3. $y = x^2\cos x \Rightarrow \frac{dy}{dx} = x^2(-\sin x) + 2x\cos x = -x^2\sin x + 2x\cos x$

4. $y = \sqrt{x}\sec x + 3 \Rightarrow \frac{dy}{dx} = \sqrt{x}\sec x\tan x + \frac{\sec x}{2\sqrt{x}} + 0 = \sqrt{x}\sec x\tan x + \frac{\sec x}{2\sqrt{x}}$

5. $y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x\cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x\cot x - \frac{2}{\sqrt{x}}$

6. $y = x^2\cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2\frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2\csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2\csc^2 x + 2x\cot x + \frac{2}{x^3}$

7. $f(x) = \sin x\tan x \Rightarrow f'(x) = \sin x\sec^2 x + \cos x\tan x = \sin x\sec^2 x + \cos x\frac{\sin x}{\cos x} = \sin x(\sec^2 x + 1)$

8. $g(x) = \csc x\cot x \Rightarrow g'(x) = \csc x(-\csc^2 x) + (-\csc x\cot x)\cot x = -\csc^3 x - \csc x\cot^2 x = -\csc x(\csc^2 x + \cot^2 x)$

9. $y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)$
$= (\sec x + \tan x)(\sec x\tan x - \sec^2 x) + (\sec x - \tan x)(\sec x\tan x + \sec^2 x)$
$= (\sec^2 x\tan x + \sec x\tan^2 x - \sec^3 x - \sec^2 x\tan x) + (\sec^2 x\tan x - \sec x\tan^2 x + \sec^3 x - \tan x\sec^2 x) = 0.$
$\left(\text{Note also that } y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.\right)$

10. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x\frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x\tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x\sin x + \cos^2 x - \cos x\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x\sec x + \cos x\sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

11. $y = \frac{\cot x}{1+\cot x} \Rightarrow \frac{dy}{dx} = \frac{(1+\cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1+\cot x)}{(1+\cot x)^2} = \frac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x\cot x + \csc^2 x\cot x}{(1+\cot x)^2} = \frac{-\csc^2 x}{(1+\cot x)^2}$

12. $y = \frac{\cos x}{1+\sin x} \Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} = \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$
$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$

13. $y = \frac{4}{\cos x} + \frac{1}{\tan x} = 4 \sec x + \cot x \Rightarrow \frac{dy}{dx} = 4 \sec x \tan x - \csc^2 x$

14. $y = \frac{\cos x}{x} + \frac{x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{x(-\sin x) - (\cos x)(1)}{x^2} + \frac{(\cos x)(1) - x(-\sin x)}{\cos^2 x} = \frac{-x \sin x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$

15. $y = x^2 \sin x + 2x \cos x - 2 \sin x \Rightarrow \frac{dy}{dx} = (x^2 \cos x + (\sin x)(2x)) + ((2x)(-\sin x) + (\cos x)(2)) - 2 \cos x$
$= x^2 \cos x + 2x \sin x - 2x \sin x + 2 \cos x - 2 \cos x = x^2 \cos x$

16. $y = x^2 \cos x - 2x \sin x - 2 \cos x \Rightarrow \frac{dy}{dx} = (x^2(-\sin x) + (\cos x)(2x)) - (2x \cos x + (\sin x)(2)) - 2(-\sin x)$
$= -x^2 \sin x + 2x \cos x - 2x \cos x - 2 \sin x + 2 \sin x = -x^2 \sin x$

17. $f(x) = x^3 \sin x \cos x \Rightarrow f'(x) = x^3 \sin x(-\sin x) + x^3 \cos x(\cos x) + 3x^2 \sin x \cos x = -x^3 \sin^2 x + x^3 \cos^2 x + 3x^2 \sin x \cos x$

18. $g(x) = (2 - x)\tan^2 x \Rightarrow g'(x) = (2 - x)(2 \tan x \sec^2 x) + (-1)\tan^2 x = 2(2 - x)\tan x \sec^2 x - \tan^2 x$
$= 2(2 - x)\tan x(\sec^2 x - \tan x)$

19. $s = \tan t - t \Rightarrow \frac{ds}{dt} = \sec^2 t - 1$

20. $s = t^2 - \sec t + 1 \Rightarrow \frac{ds}{dt} = 2t - \sec t \tan t$

21. $s = \frac{1 + \csc t}{1 - \csc t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \csc t)(-\csc t \cot t) - (1 + \csc t)(\csc t \cot t)}{(1 - \csc t)^2}$
$= \frac{-\csc t \cot t + \csc^2 t \cot t - \csc t \cot t - \csc^2 t \cot t}{(1 - \csc t)^2} = \frac{-2 \csc t \cot t}{(1 - \csc t)^2}$

22. $s = \frac{\sin t}{1 - \cos t} \Rightarrow \frac{ds}{dt} = \frac{(1 - \cos t)(\cos t) - (\sin t)(\sin t)}{(1 - \cos t)^2} = \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2} = \frac{\cos t - 1}{(1 - \cos t)^2} = -\frac{1}{1 - \cos t} = \frac{1}{\cos t - 1}$

23. $r = 4 - \theta^2 \sin \theta \Rightarrow \frac{dr}{d\theta} = -\left(\theta^2 \frac{d}{d\theta} (\sin \theta) + (\sin \theta)(2\theta)\right) = -(\theta^2 \cos \theta + 2\theta \sin \theta) = -\theta(\theta \cos \theta + 2 \sin \theta)$

24. $r = \theta \sin \theta + \cos \theta \Rightarrow \frac{dr}{d\theta} = (\theta \cos \theta + (\sin \theta)(1)) - \sin \theta = \theta \cos \theta$

25. $r = \sec \theta \csc \theta \Rightarrow \frac{dr}{d\theta} = (\sec \theta)(-\csc \theta \cot \theta) + (\csc \theta)(\sec \theta \tan \theta)$
$= \left(\frac{-1}{\cos \theta}\right)\left(\frac{1}{\sin \theta}\right)\left(\frac{\cos \theta}{\sin \theta}\right) + \left(\frac{1}{\sin \theta}\right)\left(\frac{1}{\cos \theta}\right)\left(\frac{\sin \theta}{\cos \theta}\right) = \frac{-1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \sec^2 \theta - \csc^2 \theta$

26. $r = (1 + \sec \theta) \sin \theta \Rightarrow \frac{dr}{d\theta} = (1 + \sec \theta) \cos \theta + (\sin \theta)(\sec \theta \tan \theta) = (\cos \theta + 1) + \tan^2 \theta = \cos \theta + \sec^2 \theta$

27. $p = 5 + \frac{1}{\cot q} = 5 + \tan q \Rightarrow \frac{dp}{dq} = \sec^2 q$

28. $p = (1 + \csc q) \cos q \Rightarrow \frac{dp}{dq} = (1 + \csc q)(-\sin q) + (\cos q)(-\csc q \cot q) = (-\sin q - 1) - \cot^2 q = -\sin q - \csc^2 q$

29. $p = \frac{\sin q + \cos q}{\cos q} \Rightarrow \frac{dp}{dq} = \frac{(\cos q)(\cos q - \sin q) - (\sin q + \cos q)(-\sin q)}{\cos^2 q}$
$= \frac{\cos^2 q - \cos q \sin q + \sin^2 q + \cos q \sin q}{\cos^2 q} = \frac{1}{\cos^2 q} = \sec^2 q$

30. $p = \frac{\tan q}{1 + \tan q} \Rightarrow \frac{dp}{dq} = \frac{(1 + \tan q)(\sec^2 q) - (\tan q)(\sec^2 q)}{(1 + \tan q)^2} = \frac{\sec^2 q + \tan q \sec^2 q - \tan q \sec^2 q}{(1 + \tan q)^2} = \frac{\sec^2 q}{(1 + \tan q)^2}$

31. $p = \frac{q \sin q}{q^2 - 1} \Rightarrow \frac{dp}{dq} = \frac{(q^2 - 1)(q \cos q + \sin q(1)) - (q \sin q)(2q)}{(q^2 - 1)^2} = \frac{q^3 \cos q + q^2 \sin q - q \cos q - \sin q - 2q^2 \sin q}{(q^2 - 1)^2}$
$= \frac{q^3 \cos q - q^2 \sin q - q \cos q - \sin q}{(q^2 - 1)^2}$